Dynamically optimal fertility subsidy: A policy to escape from the Empty Planet*

Wei Song†
University of Macau

Yibai Yang‡
University of Macau

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Abstract

Evidence indicates that fertility rates in many high-income countries have fallen below the replacement level, implying long-term population decline and potentially leading to the "Empty Planet" scenario. This study examines how an explicitly financed fertility subsidy can serve as a timely intervention to prevent this outcome. Our theoretical analysis characterizes the state-variant fertility subsidy that decentralizes the planner's transition toward a steady state with positive population and economic growth. Using aggregate data from Japan, we find that the current fertility subsidy rate is insufficient to reverse the negative population growth trend. Additionally, our quantitative analysis outlines the optimal fertility subsidy trajectory and suggests that the welfare loss from applying a long-run optimal fertility subsidy, rather than a dynamically first-best subsidy, depends on the economy's distance from the targeted steady state. Furthermore, while the calibrated economy's policy window will remain open for roughly long under the current subsidy, delay is nevertheless costly, with welfare losses relative to immediate adoption increasing in the length of postponement.

JEL classification: J11; J18; O30; O40
Keywords: Economic growth; Fertility; Negative population growth; Subsidy policy.

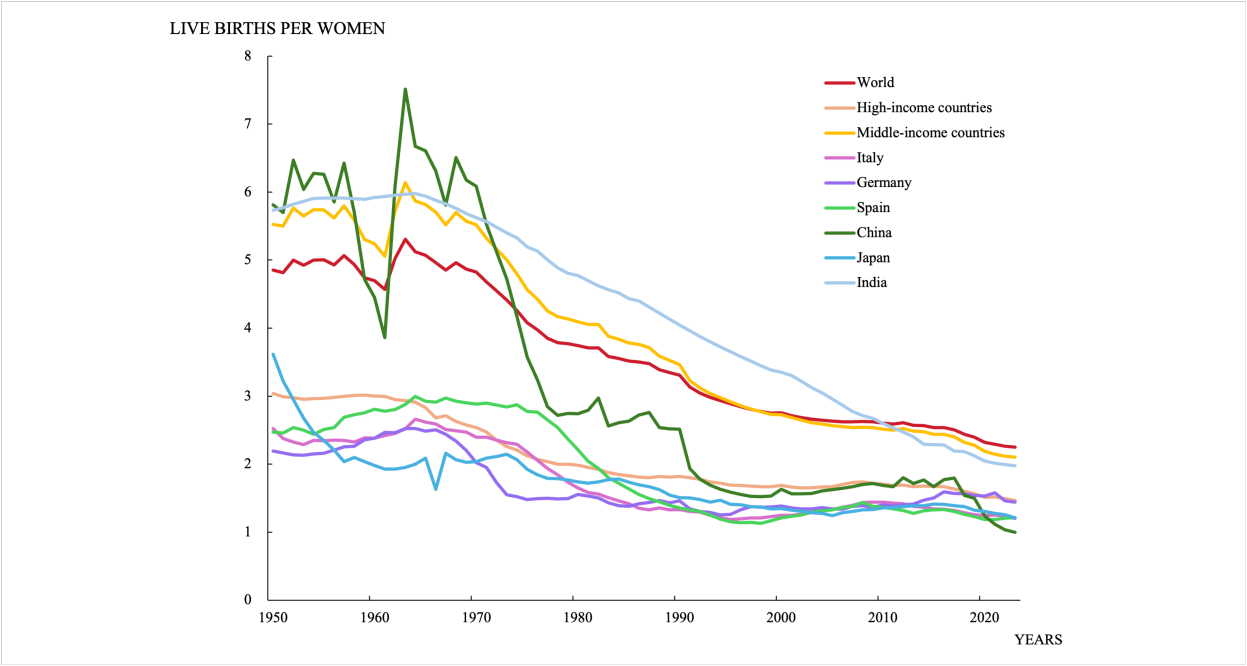
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†Department of Economics, University of Macau, Taipa, Macao, China. *Email* address: vivi_weisong@hotmail.com.
‡Department of Economics, University of Macau, Taipa, Macao, China. *Email* address: yibai.yang@hotmail.com.

1 Introduction

Over the past seven decades, many countries have experienced a significant decline in the total fertility rate (TFR). Figure 1 illustrates a sharp downward trend of the total fertility rate across various regions. Specifically, the world's average TFR has steadily decreased from 4.85 children per woman in 1950 to 2.25 in 2023. In the "reference" scenario outlined by Vollset *et al.* (2020), the global TFR is forecasted to drop to 1.66 by 2100. Since maintaining a stable population requires an average of 2.1 children per woman, known as "replacement level fertility," this low TFR of 1.66 would result in a declining population over time.

Fig. 1. Total Fertility Rate (Live Births per Women)



Note: The average number of live births refers to the number a hypothetical cohort of women would have by the end of their reproductive period, assuming that they experience fertility rates of a given period throughout their lives and are not subject to mortality. It is expressed as live births per woman. *Source: Unitied Nations, World Population Prospects 2024*

Moreover, Figure 1 demonstrates that the total fertility rates in the two most populous countries, China and India, are also declining. Notably, recent data show that China's TFR has fallen below the replacement level. High-income countries as a group, along with individual countries such as Italy, Germany, Spain, and Japan, have recorded TFRs below 2.1 in recent years as well. In the same reference scenario used by Vollset *et al.* (2020), TFRs in 183 countries are projected to fall below the replacement level by the end of this century. Additionally, 23 countries, including Japan, Thailand, and Spain, are expected to experience population declines greater than 50% from 2017 to 2100. With TFRs persistently below replacement levels, as noted by Jones (2022), forecasts – especially for high-income economies – suggest that "as countries get richer, fertil-

ity rates appear to decline to levels consistent not with a stable population, but rather with a declining population."

The ongoing decline of TFRs below replacement levels supports the idea that negative population growth rates may become the "norm" in the long term, particularly in developed regions. Table 1 confirms this idea, highlighting two significant facts. First, negative population growth appears to be an inevitable long-term phenomenon, with the global population projected to decline starting in the quinquennium 2075/2080. Second, the onset of negative population growth varies by region, with countries such as Italy, Germany, Spain, and Japan already experiencing population declines.

Table 1. Average Annual Rate of Population Change
(lower 95 percent prediction interval, 2020-2100)

	2020-2025	2025-2030	2030-2035	2035-2040	2045-2050	2055-2060	2065-2070	2075-2080	2085-2090	2095-2100
World	0.88	0.74	0.64	0.54	0.35	0.18	0.05	-0.07	-0.16	-0.25
High-income countries	0.24	0.16	0.09	0.01	-0.11	-0.19	-0.22	-0.21	-0.20	-0.19
Middle-income countries	0.78	0.62	0.50	0.39	0.17	-0.03	-0.18	-0.30	-0.39	-0.47
Italy	-0.35	-0.48	-0.54	-0.60	-0.88	-1.16	-1.19	-1.08	-1.08	-1.16
Germany	-0.24	-0.32	-0.36	-0.41	-0.55	-0.57	-0.52	-0.53	-0.48	-0.43
Spain	-0.22	-0.34	-0.41	-0.49	-0.72	-1.05	-1.22	-1.05	-0.88	-1.00
China	0.12	-0.12	-0.26	-0.42	-0.71	-0.92	-0.99	-1.07	-1.08	-1.11
Japan	-0.56	-0.75	-0.86	-0.94	-1.00	-1.14	-1.31	-1.23	-1.16	-1.21
India	0.63	0.42	0.23	0.04	-0.29	-0.62	-0.94	-1.20	-1.43	-1.60

Source: United Nations, World Population Prospects 2019, Volume I: Comprehensive Tables (Table A.15).

Population decline can lead to economic problems through various channels. In a recent paper, Jones (2022) analyzes the consequences of negative population growth using an R&D-based semi-endogenous growth model. He finds that negative population growth rates may lead the economy toward an "Empty Planet," where living standards and economic growth stagnate.[1] Additionally, his model characterizes an optimal equilibrium where, in the long run, the economy could achieve a high steady state with sustained exponential growth in population and living standards, termed the "Expanding Cosmos." Based on endogenous population growth rates in decentralized and optimal equilibria, Jones concludes:

> Policies related to fertility may therefore determine whether we converge to an "Empty Planet" or to an "Expanding Cosmos"; they may be much more important than we have appreciated...If society waits too long to adopt good policies, the optimal allocation switches from one of sustained exponential growth in population, knowledge, and living standards to one of stagnation and an Empty Planet.

His study identifies a fertility subsidy as a possible instrument for moving the decentralized economy toward the optimal allocation, warning that delaying intervention for “too long" may

[1]See Jones (2022) for details in which output per capita stagnates when negative population growth is present in both fully endogenous growth (as in Romer 1990; Aghion and Howitt 1992; and Grossman and Helpman 1991) and semi-endogenous growth models (as in Jones 1995; Kortum 1997; and Segerstrom 1998).

trap the economy in the “Empty Planet" outcome. However, his analysis leaves the design and timing of such a policy implicit. To bridge this gap, we ask three questions. First, how much fertility rates should be subsidized and how these subsidies should adjust according to the economic state, to prevent economies with negative population growth from ultimately falling into the Empty Planet outcome? Second, how large is the welfare cost of replacing this policy with a simpler subsidy fixed at its long-run optimal level? Third, how long can policy intervention be delayed before optimal fertility subsidies can no longer prevent the Empty Planet outcome, and what are the welfare costs of postponement before this critical threshold is reached?

To answer these questions, we introduce into the decentralized economy an explicit fertility subsidy through which the government compensates households for a fraction of opportunity cost of child rearing. Following Jones (2022), we first analytically examine the equilibrium allocation and the optimal allocation of the population growth rate. Within this framework, individuals decide how much time to allocate to reproductive activities, determining the population growth rate. Without intervention, the endogenous fertility rate, or "net population growth," can be negative, depending on various parameters, potentially leading the economy toward the Empty Planet outcome.

Nevertheless, an effective fertility subsidy may alter reproductive choices by lowering the opportunity cost of children, enabling positive population growth that boosts technological progress and economic growth. In a similar environment, considering that a larger population may generate more nonrival ideas and increase future income, a social planner would prefer a higher fertility rate, achieving a stable steady state with positive population growth in the long run. By examining the transition dynamics of both decentralized and optimal equilibrium, we find that to escape the Empty Planet, the economy needs to adopt a timely, state-variant fertility policy before reaching the threshold. Furthermore, time-variant fertility subsidies would gradually close the gap between the market equilibrium and the optimal population growth rates, steering the market economy toward the first-best allocation and converging to the Expanding Cosmos in the long run.

We start with the standard assumption that the dynamic fertility subsidy is financed by a lump-sum tax in the baseline model. In addition, we consider an extended case in which the subsidy is financed by a labor income tax. By reducing the return to market activity relative to child rearing, the labor income tax causes a shift in labor from manufacturing to reproduction, leading to an additional positive effect on the fertility rate. Consequently, we show that the optimal fertility subsidy required to replicate the first-best fertility rate is smaller when financed by a labor income tax compared to a lump-sum tax.

By analyzing aggregate data from Japan, a nation experiencing negative population growth, our quantitative analysis reveals that while the current fertility subsidy rate has mitigated the decline to some extent, population growth remains negative. If the government maintains the current subsidy rate, the economy risks becoming trapped in the Empty Planet outcome. To avoid this undesirable outcome, a significantly higher fertility subsidy is necessary. We also characterize the dynamically optimal, or first-best, subsidies on fertility that would guide the economy toward the stable Expanding Cosmos. The dynamically optimal subsidy may either

decrease or increase toward its long-run level, depending on the initial condition of the economy. Under Japan's current economic conditions, the social planner should adopt decreasing subsidies to steer the economy toward the Expanding Cosmos. Conversely, when initial knowledge per capita is higher, the optimal subsidy increases over time to achieve the same outcome. Additionally, considering that policymakers may be constrained to set policy instruments at time-invariant levels (i.e, the long-run optimum) for the sake of simplicity, we examine the welfare loss from such constraints. Our findings indicate that the magnitude of welfare loss depends on the initial state of the economy: the closer the initial knowledge per capita is to the high steady state, the smaller the social welfare loss from implementing constrained time-invariant fertility subsidies. Furthermore, we quantify the welfare cost of policy postponement. Under the current subsidy, it would take approximately 150 years for the Japanese economy to reach the state corresponding to Jones's notion of waiting "too long." Crucially, this horizon does not imply that delay is costless: welfare losses relative to immediate adoption rise with the length of postponement. These quantitative results remain robust under alternative knowledge production parameters.

2 Literature review

Traditional economic growth theories typically assume a constant or growing population (e.g., Romer 1990; Aghion and Howitt 1992; Jones 1995; Kortum 1997; Segerstrom 1998). Modeling negative growth is complex, as it involves intricate adjustments beyond simply replacing a positive value with a negative one. Despite this challenge, some theoretical papers address economic growth with negative population growth. Ferrara (2011) analyzes the AK model with exogenous negative population growth, emphasizing the role of capital depreciation in convergence. Christiaans (2011) investigates the impact of negative population growth on per capita income growth using a Solow model with a production function exhibiting increasing returns to scale. He finds a non-monotonic relationship between population growth rate and long-run per capita income growth, depending on the magnitude of negative population growth. Specifically, if the negative growth rate falls below a certain threshold (determined by the capital depreciation rate), the economy can exhibit positive per capita income growth.

Considering endogenous technological progress, Sasaki and Hoshida (2017) study negative population growth in a modified semi-endogenous R&D growth model, assuming zero capital depreciation. They show that the growth of technological progress declines to zero as the labor share in the R&D sector approaches zero. However, per capita output growth can remain positive due to rising capital per person. This can occur even without technological progress, as a declining population implies that aggregate capital is distributed among fewer individuals, increasing per capita income. Nonetheless, the exponential growth in capital per person predicted by Sasaki and Hoshida (2017) appears implausible empirically (Jones 2022), as the population decline rate is smaller than the depreciation rate, allowing for a steady state with constant capital per person, similar to a standard Solow model.

All the aforementioned papers conclude that with a negative population growth rate, long-run per capita income growth can be positive. In contrast, Jones (2022) presents a profound

implication: rather than continued exponential growth, living standards stagnate for a dwindling population, a phenomenon known as the "Empty Planet" result. The intuition behind this is that a smaller population means fewer researchers, resulting in fewer new ideas and lower living standards. Our paper is most closely related to Jones (2022) and complements his analysis by examining how the socially optimal allocation can be implemented through an explicitly financed fertility subsidy in an economy at risk of the Empty Planet outcome. We further quantify the welfare costs of two departures from this policy: fixing the subsidy at its long-run optimal level and postponing its adoption while the economy remains within its policy window.

Subsequently, several studies, including those by Boikos *et al.* (2023), Bucci (2023), and Strulik (2024), argue that a negative rate of population growth can sustain positive economic growth in the long run. In their analysis, a crucial mechanism prevents the economy from ending up with the Empty Planet outcome: ideas are generated through a reproducible input – human capital – rather than raw labor, and the accumulation of human capital can offset the negative effects of a declining population. In contrast to their focus, our study relies on the use of a policy instrument (i.e., dynamic fertility subsidies) to prevent the economy from falling into the Empty Planet outcome.

In addition, this study closely relates to Grossmann *et al.* (2013), who characterize dynamically optimal capital and research subsidies based on a calibrated semi-endogenous R&D growth model. Their findings indicate that while the optimal R&D subsidy is time-dependent, it exhibits minimal changes over time, even when the economy is far from its socially optimal steady state. Consequently, the welfare loss from implementing a constrained long-run R&D subsidy rate instead of the dynamically optimal policy is relatively small, reinforcing the feasibility of long-run optimal R&D subsidization as a policy recommendation. Inspired by their work, this paper investigates the time path of first-best fertility subsidies in a semi-endogenous growth model with negative population growth. Our quantitative analysis reveals that the optimal fertility subsidy is time-variant and adjusts monotonically towards the steady state, along its trajectory – whether increasing or decreasing – depending on the initial states at which an economy starts. However, unlike the minimal variation observed in optimal research subsidies, the range of variation in optimal fertility subsidy rates is significant. As a result, the magnitude of welfare loss from adopting a constrained time-invariant fertility policy is heavily influenced by how far the economy is from its socially optimal steady state, namely Expanding Cosmos. To the best of our knowledge, this is the first theoretical study that analyzes the time path of the first-best fertility subsidy in a dynamic general equilibrium framework with negative population growth.

This study also relates to Futagami and Hori (2010), who argue that the number of children in market equilibrium exceeds the counterpart in the social optimum due to the capital dilution effect of newborns. This implies that the government should not intervene in fertility choice by offering subsidies.[2] In contrast, our analysis reveals that the fertility rate in a decentralized equilibrium is below the socially optimal rate. The intuition behind this difference is that a larger population increases labor inputs for R&D activities, thereby accelerating technological progress.

[2]The return rate of capital in the social optimum is larger than in the market equilibrium. Therefore, the cost of having a child, due to the capital dilution effect, is greater in the socially optimal allocation.

Individuals do not consider this positive effect when deciding on the number of children, resulting in a fertility rate that may fall short of the socially optimal level. Therefore, our study emphasizes the necessity of policy interventions to better align individual fertility decisions with the socially optimal level.

Some developed countries facing low fertility rates have been providing cash allowances or child benefits to reduce the direct or indirect (opportunity) cost of children; 34 of 35 OECD countries offer such programs.[3] The impact of child benefits on fertility has received international attention. For instance, Milligan (2005) identifies a strong positive fertility effect of tax allowances in Quebec, estimating an increase in fertility of up to 25%. González (2013) evaluates the effects of a €2500 universal child benefit introduced in Spain in 2007, finding a 6% increase in the annual number of births. Cohen *et al.* (2013) examine changes in marginal child subsidies for third or higher births in Israel, finding a significant fertility response among lower-income families. Laroque and Salanié (2014) reports a significant effect on fertility decisions in France, and Riphahn and Wiynck (2017) find that increased child benefits in Germany lead to a fertility rise of between 9.6% and 22.6% for second births among high-income parents. These empirical studies provide evidence for the effectiveness of fertility subsidy, supporting our modeling of fertility subsidies aimed at increasing birth rates. Furthermore, our theoretical discussion on fertility subsidies in the context of negative population growth complements this body of literature.

The rest of this study is organized as follows. Section 3 describes the baseline model and analyzes the equilibrium and optimal allocations. In Section 4, we first discuss the optimal path of the fertility subsidy and then quntify the welfare costs of fixing the subsidy at its long-run optimal level and delaying adoption of the dynamically optimal policy. Section 5 alternatively allows for labor income tax. The last section concludes.

3 The baseline model

To explore the role of fertility subsidies in escaping the *Empty Planet* outcome, we incorporate a demographic policy instrument – fertility subsidy – into the semi-endogenous growth model with declining population. The model setup follows that of Jones (2022), with one modification: the government levies a lump-sum tax on household members and compensates them for a fraction of their child-rearing opportunity costs, which consist of the foregone wages of parents who leave their jobs to raise their children.

[3]Data source: OECD Family Database 2024 - PF1.3 Family cash benefits.

3.1 Households and fertility

Suppose that individuals care about consumption and their descendants. The dynastic utility function of the representative initial generation is given by [4]

$$U_0 = \int_0^\infty e^{-\rho t} \left(\ln c_t + \epsilon \ln N_t\right) dt, \quad N_0 = 1 \tag{1}$$

where c_t is consumption and N_t denotes the population size. The parameter $\rho \equiv \tilde{\rho} + \delta > 0$ is the overall discount rate that includes the rate of time preference and death rate, and $\epsilon > 0$ determines people's preference for their offspring relative to consumption.

Each person is endowed with one unit of time to allocate between child-rearing and work, with time spent on each being l_t and $1 - l_t$, respectively. Assume that devoting l_t generates a fertility rate of bl_t, and with a constant death rate δ, the population growth rate is $n_t = bl_t - \delta$. Therefore, the total number of offspring produced in an economy at time t is given by

$$\dot{N}_t = (bl_t - \delta)N_t. \tag{2}$$

Moreover, individuals face a budget constraint given by

$$c_t = w_t(1 - l_t) + s_t w_t l_t - \tau_t, \tag{3}$$

where w_t is the wage rate that individuals earn per unit of time devoted to the market sector, and τ_t is the per capita lump-sum tax collected by the government. The policy variable $s_t \in (0, 1)$ is the time-variant subsidy rate on fertility, implying that the government subsidizes a fraction s_t of the opportunity cost of child-rearing to encourage fertility.[5]

3.2 Production and ideas

Following Jones (2022), we consider a simplified version of the idea-driven semi-endogenous growth model in which all people create ideas. Using people and nonrivalrous ideas, the consumption-output good is produced according to

$$Y_t = A_t^\sigma (1 - l_t)N_t, \tag{4}$$

[4]The welfare criteria are ambiguous in the context of endogenous fertility due to significant intergenerational and intragenerational welfare implications (see discussions in Nerlove *et al.* 1986 and Golosov *et al.* 2007). In this paper, we use the welfare of a dynastic family as the criterion, which is commonly considered in the existing literature. In addition, since our analysis focuses on policy implementation along the transition path, we follow Jones (2022) in adopting log utility for the analysis of transition dynamics, although he considers more general preferences in the steady-state analysis.

[5]The representation of subsidy policy is abstract. However, according to the neoclassical economic theory of fertility, rational fertility decision making depends on the economic costs and benefits of children. Any reduction in the cost of children (due to public subsidy) or any increase in income (due to transfer payments) is expected to raise the demand for children, as suggested by Cigno (1991) and Becker (1993).

where A_t is the stock of ideas in the economy, and the parameter σ captures the degree of increasing returns. Ideas evolve according to

$$\dot{A}_t = N_t^{\lambda} A_t^{1-\beta}, \tag{5}$$

where $\lambda \in (0, 1]$ captures the stepping-on-toes effect, and $\beta \neq 1$ allows for both positive and negative externalities in research. Specifically, the case $\beta \in (0, 1)$ suggests a standing-on-shoulders effect, whereas the case $\beta > 1$ suggests an increasing-complexity effect.

3.3 Government

The government collects a lump-sum tax to subsidize child-rearing, and its budget constraint in per capita term is given by $\tau_t = s w_t l_t$.

3.4 Competitive equilibrium with externalities

Following Jones (2022), we consider a simple equilibrium in which the production of ideas is purely external. Assuming that markets are perfectly competitive, the only cost for final-good production firms is the wage $w_t = A_t^{\sigma}$ paid to workers. For the household, people do not internalize the economic benefit of fertility, that is, their children may generate more ideas that make the entire economy more productive in the future. Therefore, taking w_t as given, each household maximizes the utility function (1) subject to the population and budget constraints (e.g., equations (2) and (3)). The Hamiltonian function is

$$H_t = \ln c_t + \epsilon \ln N_t + v_t[(b l_t - \delta) N_t] + z_t[w_t(1 - l_t) + s_t w_t l_t - \tau_t - c_t], \tag{6}$$

where v_t is the co-state variable associated with the law of motion for population in (2), and z_t is the multiplier on the budget constraint in equation (3). The first-order condition with respect to l_t yields the equilibrium allocation to fertility such that

$$l_t = 1 - \frac{1 - s_t}{b V_t}, \tag{7}$$

where $V_t = v_t N_t$ captures the shadow value of the entire population. From the first-order condition with respect to N_t, we derive the law of motion for V_t such that

$$\dot{V}_t = \rho V_t - \epsilon, \tag{8}$$

which implies that $V_t = V_{eq}^* = \epsilon / \rho$ holds at all times; otherwise, the transversality condition will be violated. Combining equations (7) and (8), the population growth rate is given by

$$n_{eq}^* = b - \frac{\rho}{\epsilon} - \delta + s_t \frac{\rho}{\epsilon}, \tag{9}$$

implying that implementing a fertility subsidy s_t raises the fertility rate by $s_t\rho/\epsilon$. Moreover, in the absence of a fertility subsidy (i.e., $s_t = 0$), the equilibrium features a constant population growth rate over time, since the shadow value of people is constant (i.e., $V_t = V_{eq}^* = \epsilon/\rho$). In this case, the equilibrium population growth rate without a fertility subsidy is given by the "net population growth rate," which can be positive or negative depending on parameter values. The negative case, which is the focus of this study, would result in the Empty Planet outcome, where living standards stagnate as the population falls at a constant rate. In the next subsection, we consider the optimal allocation in this model to explore the dynamics for optimal fertility subsidies.

3.5 The optimal allocation

Different from the above equilibrium allocation, we follow Jones (2022) to consider the optimal allocation that takes into account the extra benefit associated with increasing returns to scale. That is, larger population creates more nonrival ideas that benefit all agents in the economy. To maximize the welfare of a representative generation, the social planner solves

$$\max_{\{l_t\}} \int_0^\infty e^{-\rho t} \left(\ln c_t + \epsilon \ln N_t\right) dt,$$

subject to

$$\begin{aligned} \dot{N}_t &= \left(bl_t - \delta\right) N_t, \\ \frac{\dot{A}_t}{A_t} &= N_t^\lambda A_t^{-\beta}, \\ c_t &= Y_t/N_t = A_t^\sigma \left(1 - l_t\right). \end{aligned}$$

The Hamiltonian for this problem is

$$H = \ln c_t + \epsilon \ln N_t + v_t \left(bl_t - \delta\right) N_t + \mu_t N_t^\lambda A_t^{1-\beta},$$

where v_t is the co-state variable associated with the law of motion in equation (2), capturing the shadow price of an additional person. The variable μ_t is the co-state variable associated with the law of motion in equation (5), capturing the shadow price of an idea. The first-order condition with respect to l_t is

$$l_t = 1 - \frac{1}{bV_t}, \tag{10}$$

which implies that the population growth rate is

$$n_t = b - \delta - \frac{1}{V_t}, \tag{11}$$

The first-order condition with respect to A_t can be expressed as:

$$\rho = \frac{\dot{\mu}_t}{\mu_t} + \frac{1}{\mu_t}\left[\sigma \frac{1}{A_t} + \mu_t(1-\beta)\frac{\dot{A}_t}{A_t}\right]. \tag{12}$$

Define a new variable $z_t \equiv \mu_t \dot{A}_t$, which represents the social value of new ideas produced in period t. From equation (12), z_t is constant along the balanced growth path (BGP) and given by

$$z^* = \frac{\sigma g_A^*}{\rho + \beta g_A^*}. \tag{13}$$

Notably, if $n^* < 0$ such that $g_A^* = 0$, the social value of new ideas is zero. This means no new ideas are being produced in the Empty Planet scenario, resulting in a value of zero.

Next, the first-order condition with respect to N_t is given by

$$\rho = \frac{\dot{v}_t}{v_t} + \frac{1}{V_t}(\epsilon + \lambda \mu_t \dot{A}_t) + n, \tag{14}$$

which yields the optimal shadow value of the population along the BGP such that

$$V^* = \frac{\epsilon + \lambda z^*}{\rho}. \tag{15}$$

Then the steady-state population growth is given by

$$n^* = b - \delta - \frac{1}{V^*}, \tag{16}$$

Therefore, equations (13), (15), and (16) determine the steady-state values for z^*, V^*, and n^*. As shown in Jones (2022), the steady-state growth rate of A_t is given by

$$g_A^* = \begin{cases} \frac{\lambda}{\beta} n^* & if \quad n^* > 0 \\ 0 & if \quad n^* \leq 0 \end{cases}$$

That is, in the semi-endogenous growth model, the steady-state growth rate of technology is proportional to n^* if population growth is positive; otherwise, it is zero.

This kink leads to the existence of multiple steady states. On the one hand, suppose that there exists a negative solution $n^* < 0$ such that $g_A^* = 0$. We have $z^* = 0$ and $V^* = \epsilon/\rho$, implying $n_{sp}^* = b - \delta - \rho/\epsilon < 0$. This implies that the Empty Planet result is also a solution for the optimal allocation problem when the "net population growth rate" is negative. On the other hand, suppose that there exist positive solutions $n^* > 0$. Equations (13), (15), and (16) yield a quadratic equation with respect to n^*:

$$(n^*)^2 - \left[b - \delta - \frac{\rho}{\lambda} \frac{\epsilon + \lambda}{\epsilon + \gamma} \right] n^* - \frac{\rho\epsilon(b - \delta - \rho/\epsilon)}{\lambda(\epsilon + \gamma)} = 0, \tag{17}$$

where $\gamma \equiv \lambda\sigma/\beta$. Assume that the overall degree of increasing returns γ is sufficiently large such that $B \equiv b - \delta - (\rho/\lambda)[(\epsilon + \lambda)/(\epsilon + \gamma)] > 0$. Then the quadratic function has only one positive root when the "net population growth rate" is positive (i.e., $b - \delta - \rho/\epsilon > 0$) such that $C \equiv -[\rho\epsilon(b - \delta - \rho/\epsilon)]/[\lambda(\epsilon + \gamma)] < 0$. Conversely, it has two positive roots if the "net population

growth rate" is negative (i.e., $b - \delta - \rho/\epsilon < 0$) such that

$$n^*_{sp\{h,m\}} = \frac{B \pm \sqrt{B^2 - 4C}}{2}, \tag{18}$$

where $n^*_{sp\{h\}}$ corresponds to the higher positive population growth rate and $n^*_{sp\{m\}}$ corresponds to the lower one.

In summary, when the equilibrium population growth is negative, the optimal allocation features three steady states: (1) an "Expanding Cosmos" steady state with positive population growth; (2) a middle steady state also with positive population growth; (3) an "Empty Planet" steady state with negative population growth. In the next subsection, we will discuss the transitional dynamics for the optimal allocation and the optimal fertility subsidy.

3.6 Optimal fertility subsidy

To explore the optimal fertility subsidy, we first analyze the transitional dynamics of the optimal allocation. Define two new state variables $p_t \equiv \ln N_t$ and $x_t \equiv A_t^\beta / N_t^\lambda$, with the latter referred to as "knowledge per person." Notably, the state variable x_t is also the inverse of the growth rate of knowledge (i,e,. $\dot{A}_t / A_t = 1/x_t$). Following Jones (2022), the dynamical system of the optimal allocation is determined by

$$\dot{x}_t = \beta - \lambda n_t x_t, \tag{19}$$

$$\dot{n}_t = -\left(\bar{b} - \delta - n_t\right)^2 \left[\left(\rho + \frac{\beta}{x_t}\right)\left(\nu - \frac{1}{\bar{b} - \delta - n_t}\right) - \gamma\right], \tag{20}$$

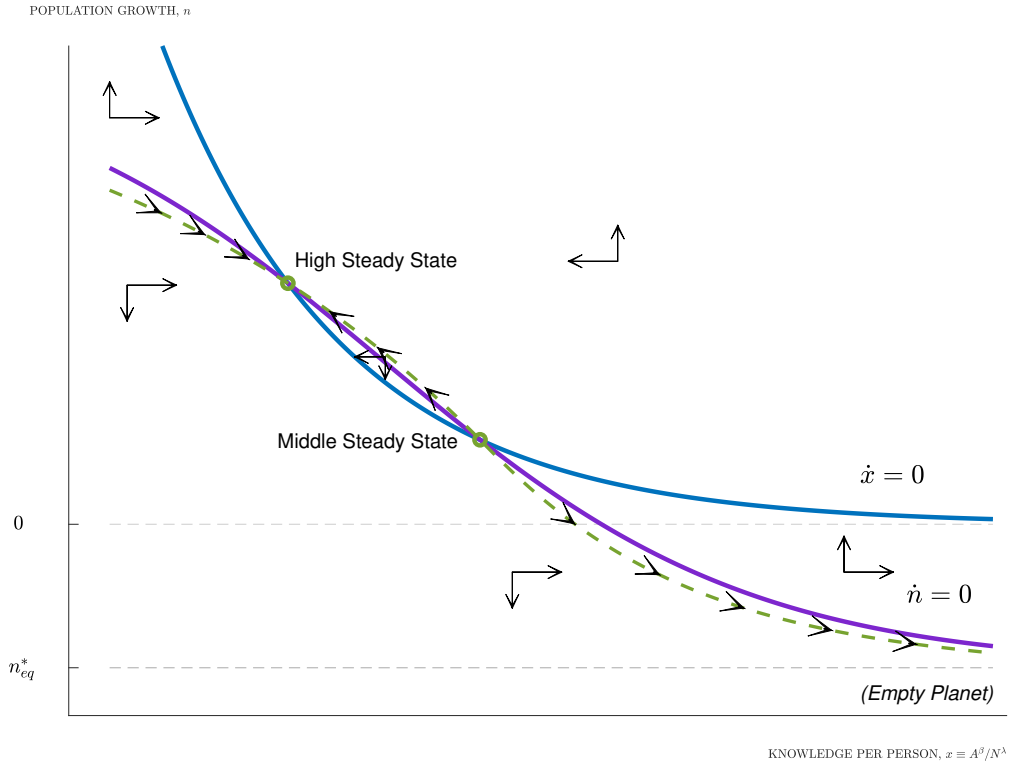
where $\nu \equiv (\epsilon + \gamma)/\rho$. Combined with an initial condition x_0 and a transversality condition, these two differential equations define the optimal path for population growth. Figure 2 shows the phase diagram and the path for the optimal allocation when the equilibrium features negative population growth (i.e., $b - \delta - \rho/\epsilon < 0$).

As shown in Figure 2, there are two interior steady states, corresponding to the high steady state and the middle steady state. The high steady state is stable, whereas the middle steady state is unstable. Moreover, the "Empty Planet" steady state is also stable. For values of x smaller than the middle steady state, the dynamics lead the economy to the "Expanding Cosmos" steady state; otherwise, it leads the economy to the Empty Planet steady state.

When the equilibrium features negative population growth, the economy simply evolves to the Empty Planet. In this case, if a fertility subsidy that decentralizes the socially optimal allocation is implemented before knowledge per person reaches the unstable middle steady-state threshold x^*_m, it places the economy on the planner's path toward the Expanding Cosmos. Conversely, if optimal policy adoption is delayed until knowledge per person has crossed this threshold, the economy would be lead toward the Empty Planet.[6] This policy implication is summa-

[6]Notably, we do not rule out alternative, suboptimal fertility policies that could steer the economy toward the high steady state, since fertility can jump upon policy adoption. The threshold concerns the planner's optimal choice rather than the technological feasibility of reaching the high steady state.

Fig. 2. The Phase Diagram for the Optimal Allocation



Note: The dashed green line shows the optimal path for the population growth in the (x, n) space.

rized in the following result.

Result 1. *To escape the Empty Planet, the social planner can adopt an optimal fertility subsidy in a timely manner in the presence of negative population growth. That is, the optimal fertility subsidy must be adopted before the knowledge per capita x_t reaches the threshold $x^*_m = \sigma/(\gamma n^*_{sp\{m\}})$. Once this threshold has been crossed, implementing the subsidy that decentralizes the socially optimal allocation cannot steer the economy away the Empty Planet.*

The preceding result identifies the states from which the economy can escape the Empty Planet along the socially optimal path. For an economy that remains within the policy window ($x_t < x^*_m$), the following proposition characterizes a state-variant fertility subsidy s_t that can lead the economy to attain the first-best optimum.

Proposition 1. *Under lump-sum taxation, the optimal path of population growth can be supported by setting a time-variant fertility subsidy s_t, where the optimal fertility subsidy, denoted as s^{opt}_t, satisfies:*

$$n_t = b - \frac{\rho}{\epsilon} - \delta + s^{opt}_t \frac{\rho}{\epsilon}, \tag{21}$$

where n_t evolve according to equations (19) and (20). Moreover, the long-run optimal fertility subsidy is

given by

$$(s^{opt})^* \equiv s_h^{opt} = \frac{\epsilon}{\rho}[(n^*_{sp\{h\}} - (b - \rho/\epsilon - \delta)].$$

Proof. Equations (19) and (20) illustrate the dynamics of the optimal allocation in this economy, where the social planner takes into account the positive effect of population size on R&D activities. The externality of the "idea value of people" drives, in the absence of policy intervention, a wedge between the decentralized equilibrium and the social optimum, calling for the first-best fertility subsidy. Nevertheless, given that the paths of population growth in both the decentralized equilibrium and social optimum are determined by the state of knowledge per capita x_t, the first-best fertility subsidy rate must be dynamically optimal. Finally, equation (21) portrays the fertility rate in the decentralized equilibrium under time-varying fertility subsidies, which exactly aligns with the optimal fertility rate. □

Due to the complexity of the dynamics of the optimal allocation, it is theoretically challenging to obtain the analytical optimal path for the fertility subsidy. In the next section, we will provide a quantitative analysis to examine the dynamics of the optimal fertility subsidy.

4 Quantitative analysis

In this section, we first calibrate the model and quantitatively solve for the transition dynamics for both the equilibrium and optimal allocations. Then we explore the optimal path for the fertility subsidy. We also compare the welfare differences under the first-best (i.e., state-varying) fertility subsidy and the optimal long-run (i.e., state-invariant) fertility subsidy. Subsequently, we quantify the welfare loss of delaying the first-best fertility policy.

4.1 Calibration and dynamics

We calibrate our model to the Japanese economy, which has been experiencing negative population growth since 2011, and therefore provides an important baseline for discussing how fertility policies are related to negative population growth. The model features the following set of parameters $\{\rho, \delta, \sigma, \lambda, \beta, \epsilon, b, s\}$. Following Jones (2022), the value of σ is normalized to unity so that A has the units of total factor productivity. The degree of returns to research λ and the extent of research externality β are set to 0.75 and 2, respectively. Additionally, the death rate δ is set at 1.1% to match the average death rate of Japan in the last decade.[7] Thus, the discount rate is set as $\rho = \delta = 1.1\%$. We set the fertility subsidy rate s at 2.3% to match the family cash benefit per child.[8] Then we calibrate the parameters $\{\epsilon, b\}$ to match the following empirical moments: (1) the population growth rate, and (2) the fraction of time spent on child-rearing. For (1), we follow Jones (2022) to set the population growth rate at -0.5%. For (2), according to the Japanese

[7] Data Source: World Development Indicators.

[8] According to the OECD Family Database, the value of total family benefits for a two-parent, two-earner family with a youngest child aged 6 is expressed as 2.3% of average full-time earnings (AW).

Survey on Time Use and Leisure Activities in 2021, an average person in households with children under 6 spends about 2 hours per day on childcare. Based on an average of 16 hours of non-sleeping time per day, the fraction of time spent on childcare is approximately 1/8. Table 2 summarizes these targeted moments and calibrated parameter values.

Table 2: Benchmark Calibration Summary

Parameters	Description	Value	Source/Target
External			
σ	Degree of increasing returns	1	Normalized
λ	Duplication externalities	0.75	Jones (2022)
β	Spillover effects of research	2	Jones (2022)
Internal			
δ	Death rate	1.1%	Average death rate (1.1%)
ρ	Overall discount rate	1.1%	$\rho = \delta$
s	Subsidy rate	2.3%	Cash benefit per child
ϵ	Preference parameter	0.256	Population growth rate (-0.5%)
b	Productivity of reproduction	0.048	Time allocation on childcare (1/8)

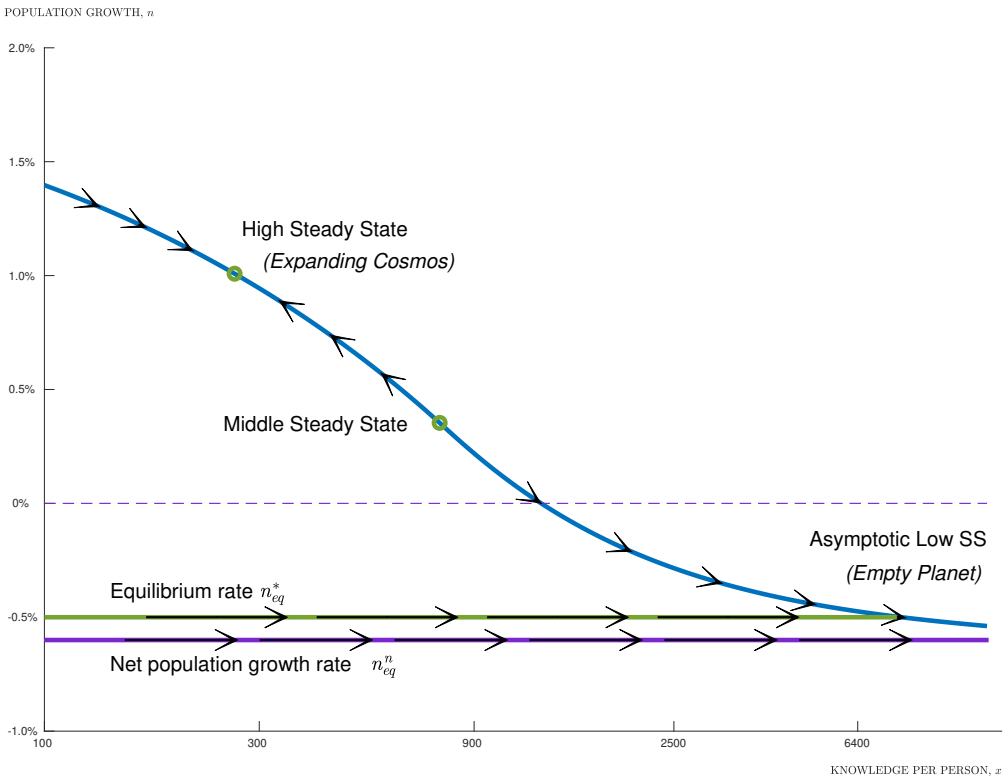
Using the aforementioned calibrated parameter values, Figure 3 summarizes the transition dynamics for the equilibrium allocation and optimal allocation when an economy features negative population growth. Assume that the Japanese economy is initially endowed with a certain population N_0 and stock of knowledge A_0 such that the knowledge per person x_0 is equal to 200, implying a TFP growth rate of $\dot{A}_0/A_0 = 1/x_0 = 0.5\%$.[9] If the government had not implemented any fertility subsidies in the past, and will not implement any in the future, the level of population would decline at a constant "net population growth rate" of $n^n_{eq} = -0.6\%$. With this constant negative population growth rate, knowledge per person $x_t \equiv A_t^\beta / N_t^\lambda$ rises over time since A_t rises to an upper bound while N_t decreases. As shown by the bottom purple line, the knowledge per person x_t eventually goes to infinity so that the growth rate of technology reduces to zero, indicating the Empty Planet result. In other words, if the population continues to decline without any intervention (in terms of fertility subsidy), the Japanese economy would eventually converge to the Empty Planet scenario.

However, to promote fertility and mitigate negative population growth, the government has implemented a fertility subsidy at the rate of 2.3%. Although the equilibrium population growth rate has risen, it remains negative ($n^*_{eq} = -0.5\%$). If the government maintains the current fertility policy and fails to promptly implement the optimal fertility subsidy, the economy will continue along the green trajectory, even it switches to the optimal allocation when the knowledge per person reaches $x_t = 7816$. In this case, the economy would also get trapped in the Empty Planet outcome: living standards stagnate for a dwindling population. To avoid this undesirable

[9] According to PWT10, the average growth rate of total factor productivity in Japan from 2010-2019 is 0.5%.

situation, a minimum fertility subsidy rate of 13.96% is required to ensure that the equilibrium population growth rate turns positive.[10]

Fig. 3. Transition Dynamics for Equilibrium Allocation and Optimal Allocation



Note: The curved blue line shows the transitional dynamics for the optimal allocation, the green line shows the dynamics for the equilibrium allocation with a constant fertility subsidy rate 2.3% and the bottom purple line shows the dynamics for the "net population growth" without any fertility subsidy.

Recognizing that larger population generates more nonrival ideas that enhance overall output, the social planner would prefer a higher fertility rate than the equilibrium allocation. Additionally, the social planner aims to steer the economy to converge to the Expanding Cosmos steady state, where both population and per capita income rise perpetually in an exponential fashion. To achieve this, the government should implement fertility policies that incentivize households to have more children, driving the economy toward the Expanding Cosmos rather than getting trapped in the Empty Planet. As illustrated by the blue curved line and the purple line, the government needs to implement dynamically optimal fertility subsidies to bridge the gap between the net population growth rate and the optimal growth rate. This allows the equilibrium allocation to shift to the optimal allocation and converge to the Expanding Cosmos along the path of the optimal allocation. However, an important question arises: does this policy work regardless

[10]Note that that the magnitude of fertility subsidy in this quantitative exercise is not intended as a direct policy recommendation. Because these estimates are obtained under the assumption of lump-sum taxation, which abstracts from fiscal financing constraints, they serve primarily as a benchmark to highlight the insufficiency of current subsidy levels.

of when it is implemented?

To address this question, we first quantitatively analyze the transition path for the optimal allocation. As depicted by the blue curved line: for a value of x_t below the high steady state $x_h^* = 264.5$, the dynamics guide the optimal allocation to the right, so that the economy ultimately converges to the stable high steady state. Furthermore, if the economy is initially situated between the high steady state and the middle steady state (i.e., $264.5 < x_t < 754.7$), the economy will converge to the stable high steady state as well, but the movement will be from right to left. Moreover, if the value of knowledge per person is above the middle steady state (i.e., $x_t > x_m^* = 754.7$), the optimal allocation transitions to the Empty Planet steady state. Therefore, in line with Result 1, it becomes evident that the social planner must adopt optimal fertility subsidies before the knowledge per person x_t reaches the critical threshold of $x_m^* = 754.7$, where the TFP growth rate is $g_A = 0.13\%$. Failing to do so can result in the economy falling into the Empty Planet trap, even if optimal fertility subsidies are later implemented.

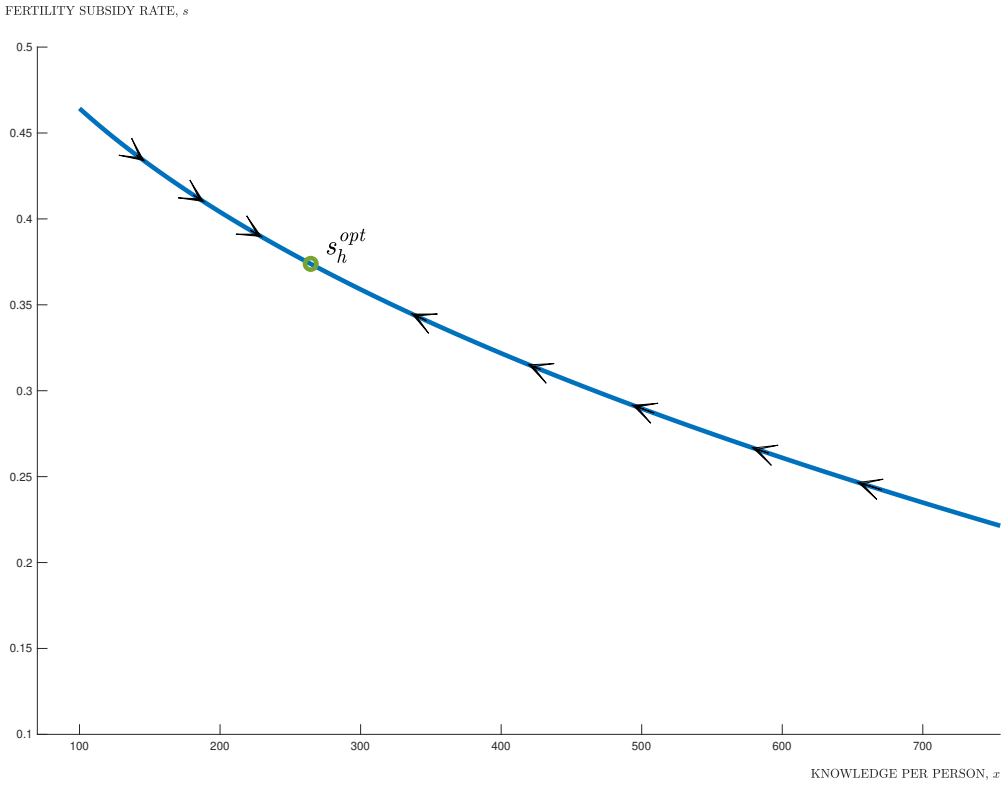
4.2 Dynamically optimal fertility subsidy policy

As previously mentioned, the implementation of fertility policy needs to follow two principles: first, timely adoption of the optimal fertility policy to prevent the economy from falling into the Empty Planet outcome; second, adoption of a state-variant optimal fertility policy so that the economy can ultimately move towards the stable Expanding Cosmos along the trajectory of the optimal allocation. Therefore, this subsection focuses on the analysis of the dynamically optimal fertility subsidy policy where the economy remains within the policy window ($x_t < 754.7$).

First, according to the dynamical system in Proposition 1, Figure 4 illustrates how the optimal fertility subsidy rate s_t^{opt} evolves with knowledge per person x_t. The high steady state optimal fertility subsidy $s_h^{opt} = 37.38\%$ is stable, indicating that implementing a 37.38% subsidy when x_t exactly reaches $x_h^* = 264.5$ will directly shift the equilibrium to the Expanding Cosmos steady state.

Moreover, it is worthwhile noting that the optimal fertility subsidy can converge to the high steady state optimal fertility subsidy rate s_h^{opt} from two directions. On the one hand, if the initial population and stock of knowledge result in a knowledge per person value x_0 below the high steady state $x_h^* = 264.5$, the optimal fertility subsidy rate starts above the long-run optimum s_h^{opt} and decreases as x_t increases. Initially, the optimal fertility subsidy rate is high due to the large gap between the net population growth rate n_{eq}^n and the optimal population growth n_{sp}^h. As x_t rises, the growth rate of knowledge declines, reducing the "knowledge value" of additional offspring. This reduction in externality implies a shrinking gap between the net population growth rate and optimal allocations, requiring a smaller fertility subsidy to close this gap. On the other hand, if the initial value of knowledge per person x_0 satisfies $x_0 \in (x_h^*, x_m^*)$, the optimal fertility subsidy increases as x_t declines, eventually converging to the long-run optimal fertility subsidy $s_h^{opt} = 37.38\%$. In other words, when the subsidy rate starts at a lower level (i.e., $s_m^{opt} = 22.15\%$) given the relatively small gap between the net population growth rate and optimal allocations, as x_t decreases, the gap widens due to the increasing externality from the *idea value of people*, necessitating a higher fertility subsidy to drive the economy towards the high steady state.

Fig. 4. Optimal Path for Fertility Subsidy Rate



Consequently, the optimal subsidy gradually increases to the long-run level of $s_h^{opt} = 37.38\%$ as the economy approaches the high steady state.

In a word, if the high steady-state optimal fertility subsidy is implemented exactly at x_h^*, the economy will directly switch to the Expanding Cosmos; otherwise, the social planner needs to adopt a dynamically optimal fertility subsidy policy to lead the economy to converge to the Expanding Cosmos along the path of the optimal allocation. In this case, there are two scenarios to consider. First, if the optimal fertility subsidy is implemented before the economy reaches the high steady-state knowledge level x_h^*, then the government should adopt decreasing optimal fertility subsidies until the rate reaches the long-run optimal level s_h^{opt}. This allows the equilibrium population growth rate to gradually converge to the high-steady-state population growth n_{sp}^h. In contrast, if the optimal fertility subsidy is implemented after the economy has passed the high steady-state knowledge level x_h^*, then the government needs to implement an increasing fertility subsidy rate to move the economy back towards the Expanding Cosmos.

The state-varying optimal fertility subsidy policy characterizes the dynamically optimal, or first-best, subsidies on child-rearing. However, one might argue that such a state-variant policy would be challenging to implement in real practice, as it could be difficult for policymakers to exactly locate the economy along the transition path. Consequently, two important questions arise. First, for simplicity, can we adopt instead a time-invariant fertility subsidy rate set at the long-run optimal level (i.e., s_h^{opt})? Second, how large is the (welfare) cost of this alternative policy

implementation? We investigate these two questions in the next subsection.

4.3 Constrained optimal policy versus first-best policy

We first quantitatively explore how the variables of interest evolve along the transition path under two policy regimes: the constrained optimal policy (i.e., the optimal high steady-state fertility subsidy rate s_h^{opt} is implemented from the start) versus the first-best policy (i.e., the dynamically optimal fertility subsidy is implemented throughout the transition).[11] By comparing the transition paths under these two policy regimes, we can gain insights into the implications of the constrained optimal policy relative to the first-best counterpart. Furthermore, we discuss the feasibility of the constrained optimal policy in terms of the magnitude of the intertemporal welfare difference between these two policy regimes. Similar to the previous subsection, we consider two cases. First, the initial knowledge per person x_0 is below the high steady-state knowledge level x_h^*, indicating that the optimal fertility subsidy rate decreases to reach the long-run optimum s_h^{opt}. Second, the initial knowledge per person x_0 is above x_h^*, suggesting that the optimal fertility subsidy rate gradually increases to converge to s_h^{opt}.

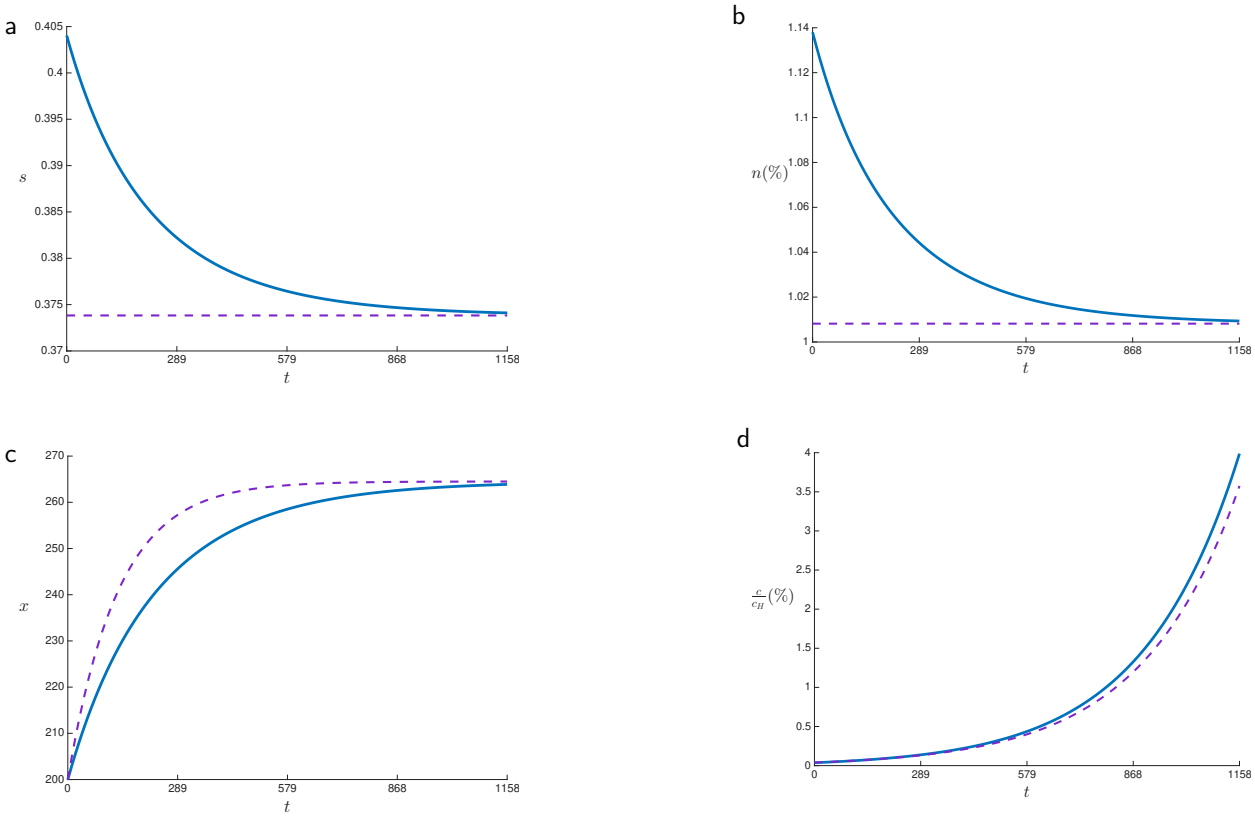
4.3.1 Low initial knowledge per person

Recalling that the knowledge per person x_0 is located at 200 such that the TFP growth rate is $\dot{A}_0/A_0 = 1/x_0 = 0.5\%$, we first examine how relevant variables evolve over time when this initial condition is in place, as illustrated in Figure 5. The blue line in panel (a) depicts the trajectory of the time-varying optimal fertility subsidy rate, s_t^{opt}, which starts above the long-run optimum, s_h^{opt}, and decreases over time. This indicates that the optimal fertility subsidy should be initially set higher due to the substantial gap between the equilibrium fertility rate and the targeted optimal level. As the fertility rate n_t approaches the high steady-state level, n_{sp}^h, the optimal subsidy rate gradually declines. Under the dynamically optimal fertility subsidy policy, the fertility rate initially jumps to 1.138% and then gradually falls to the high steady-state level of 1.01%, as shown by the blue line in panel (b). In contrast, the constrained optimal policy maintains the fertility subsidy rate fixed at the long-run optimal level, $s_h^{opt} = 37.38\%$, resulting in a constant fertility rate remaining at the high steady-state, as illustrated by the purple lines in panels (a) and (b).

Panel (c) shows that the knowledge per person x_t rises along a flatter trajectory under the first-best policy, whereas it exhibits a steeper upward trend under the time-invariant constrained optimal fertility subsidy policy and reaches the high steady-state level x_h^* at an earlier point in time. This can be attributed to the law of motion for x_t, given by $\dot{x}_t = \beta - \lambda n_t x_t$. Since the fertility rate n_t under the constrained optimal policy is lower than that under the first-best policy (as observed in panel (b)), the term $\lambda n_t x_t$ becomes smaller. Consequently, the growth rate of knowledge per person, $\dot{x}_t$, is larger under the constrained optimal policy, leading to faster convergence to the high steady-state level x_h^*. Finally, recall that the per capita consumption c_t

[11] Analogous to the analysis of R&D subsidy in Grossmann *et al.* (2013), the time-invariant fertility subsidy rate set at its long-run level is referred to as the "constrained optimal policy."

Fig. 5. Transition Dynamics for Variables of Interest (Benchmark)



Note: Scenario (i): $x_0 = 200 < x_H$. The blue solid lines: first-best solution and the purple dashed lines: constrained optimal policy.

can be expressed as $c_t = x_t^{\gamma/\lambda} N_t^{\gamma}(1 - l_t)$, which implies that c_t is determined by the level of knowledge per person x_t, the population N_t, and the labor fraction allocated to production $1 - l_t$. With the interplay of these three variables, the evolution paths of the consumption per capita c_t, under the constrained optimal policy and the first-best policy are quite similar, as depicted in panel (d).

Next, we investigate the welfare difference between implementing the dynamically optimal fertility policy and the optimal high steady-state policy, considering the transition path from the initial state where $x_0 = 200$ (see Appendix A.2 for details). The intertemporal welfare difference under these two policy regimes is expressed in terms of the equivalent variation in consumption. As shown in the second panel of Table 3, implementing the constrained optimal fertility subsidy leads to a welfare loss, indicating that the dynamically optimal fertility subsidy is a more favorable policy regime. However, the magnitude is small, approximately 0.0497%, suggesting that adopting the high steady-state fertility subsidy will not significantly harm the welfare of the Japanese economy. According to equation (1), the welfare level is determined by consumption per capita c_t and the population N_t. The welfare difference due to consumption per capita c_t is 1.883, whereas the difference related to the population N_t is -1.928. These two offsetting effects

contribute to the small overall welfare loss, which aligns with the intuition. Although the social planner attempts to reduce the optimal fertility subsidy rate as the fertility gap narrows to improve welfare, given that the initial dynamically optimal fertility subsidy rate (i.e., 40.4%) is quantitatively close to the high steady-state value of 37.38%, the welfare gain of moving from the constrained optimal policy to the first-best policy is not significant.

Table 3: Steady State Values and Welfare Analysis (Benchmark)

Panel A. Steady-state values				
	n_{sp}	x	g_A	s^{opt}
High steady state	1.01%	264.5	0.38%	37.38%
Middle steady state	0.35%	754.7	0.13%	22.15%
Panel B. Welfare loss from implementing the constrained optimal policy				
Initial condition	(i) $x_0 = 200$	(ii) $x_0 = 100$	(iii) $x_0 = \frac{x_h + x_m}{2}$	(iv) $x_0 = \lfloor x_m - 0.1 \rfloor$
Welfare loss	0.0497%	0.4511%	0.3829%	1.1021%
Panel C. Welfare cost of delayed policy adoption ($T_{\text{window}} \approx 150$)				
Delayed years	(i) $t = 30$	(ii) $t = 60$	(iii) $t = 90$	(iv) $t = 120$
Welfare cost	3.4803%	5.2790%	6.2152%	6.6972%

We also consider an alternative scenario where the initial knowledge per person x_0 is set at 100. In this case, the welfare loss increases to 0.4511%, indicating a larger impact than in the previous scenario. This larger welfare loss is associated with a greater variation in the dynamically optimal fertility subsidy rate and the high steady-state optimal fertility subsidy.

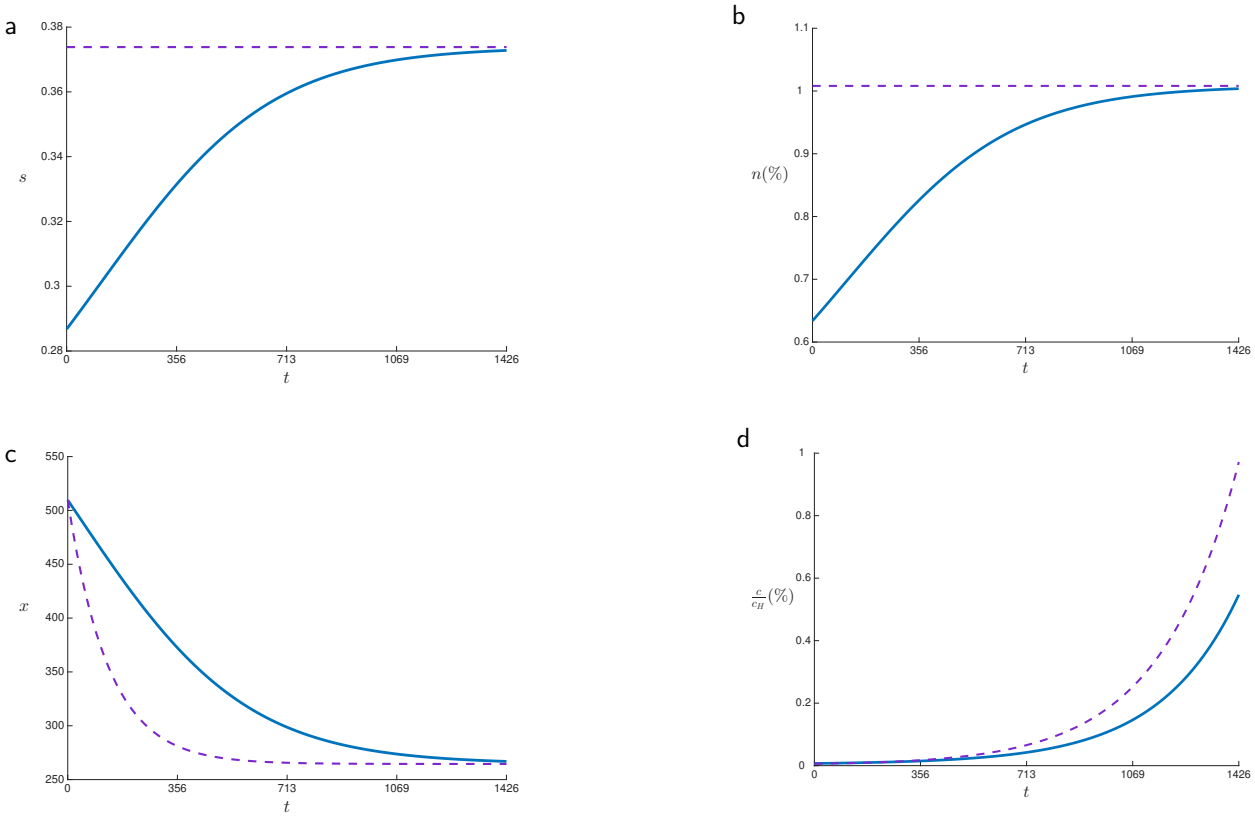
In summary, the welfare cost from choosing the constrained optimal fertility policy instead of the first-best policy is closely tied to the initial conditions. Specifically, the social planner may implement a time-invariant high steady-state optimal fertility subsidy only when the knowledge per person is relatively close to that steady-state level; the entire economy will not pay a large cost in this case. Otherwise, the constrained fertility subsidy policy can readily lead to a significant detriment to social welfare.

4.3.2 High initial knowledge per person

Next, we consider the case in which the economy begins on the right-hand side of the "Expanding Cosmos." We first examine a scenario where the initial knowledge per person is located at the midpoint between the high and middle steady states, set at $x_0 = 509.6$. Figure 6 shows the dynamic trajectory of the key variables.

In this case, the dynamically optimal fertility subsidy rate, depicted by the blue line in panel (a), gradually increases to its long-run optimum. The associated fertility rate n_t follows the same

Fig. 6. Transition Dynamics for Variables of Interest (Benchmark)



Note: Scenario (iii): $x_0 = 509.6 > x_h$. The blue solid lines: first-best solution and the purple dashed lines: constrained optimal policy.

pattern in panel (b). By contrast, under the constrained policy, the subsidy and fertility rate are fixed at their high steady state levels and therefore remain above their first-best counterparts throughout the transition. Given $\dot{x}_t = \beta - \lambda n_t x_t$, the higher fertility rate causes knowledge per person to decline more rapidly under the constrained policy, resulting in the separation between the two paths in panel (c). As for the dynamics of consumption per capita c_t, the first-best solution increases over time in a flatter trajectory, whereas the solution under the constrained optimal high steady-state fertility subsidy first rises slowly and then surges, causing a divergence in consumption per capita under these two policy regimes.

We now examine the welfare difference arising from the implementation of the constrained optimal fertility subsidy in this scenario. Unlike the previous scenario, the contribution of population to the intertemporal welfare difference becomes positive (i.e., 6.860), but it is outweighed by the negative contribution from consumption (i.e., -7.209), leading to a welfare loss of 0.3829%. This non-negligible welfare loss suggests that the constrained optimal policy may not be desirable if the economy is located at the midpoint between x_h^* and x_m^*. Moreover, as the economy moves further away from x_h^* toward x_m^*, the divergence between the fixed subsidy and the dynamically optimal subsidy path widens, significantly raising the welfare loss to 1.1021%. These

findings suggest that once the economy moves beyond the high steady state (i.e., when knowledge per person x_t reaches the level corresponding to the high steady state x_h^*), further delays in policy intervention lead to increasingly large welfare losses under the constrained optimal policy. The social planner may therefore need to adopt a more complex policy regime involving dynamically optimal fertility subsidies that are time-varying. The alternative simplified constrained optimal policy is appropriate only when the social planner intervenes sufficiently early, while the economy remains close to x_h^*.

4.4 Welfare cost of delayed policy adoption

While the previous analysis focuses on policy design by assessing whether a time-invariant subsidy can replace the dynamically optimal policy, we now turn to the dimension of policy timing. Regarding the policy timing, Jones (2022) emphasizes that long delays of good policy can shift the long-run allocation selected by the social planner to the undesirable outcome, with the state boundary for this shift identified in Result 1. However, the threshold result does not reveal whether postponement is costly while the long-run destination remains unchanged. Accordingly, this subsection first estimates the time horizon remaining before Japan reaches the state boundary under its current policy regime. Subsequently, we assess the welfare costs associated with delaying policy implementation while the economy remains within the policy window.

Given the initial condition $x_0 = 200$, the Japanese economy reaches the threshold $x_m^* = 754.7$ in approximately 150 years if the current fertility subsidy rate of 2.3% remains unchanged.[12] With this time window, we then compare immediate implementation of the dynamically optimal subsidy with adoption after delays of 30, 60, 90, and 120 years.[13] During each delay, the current policy remains in place and the population declines at 0.5 percent each year. At the adoption date, the government switches to the dynamically optimal subsidy that decentralizes the economy to the path toward the Expanding Cosmos. Welfare includes both utility during the delay and the continuation value along the first-best path after adoption. We measure the resulting consumption-equivalent welfare loss relative to immediate implementation. As shown in the third panel of Table 3, the welfare cost increases with the length of the delay. Delaying adoption by 30 years generates a consumption-equivalent loss of 3.4803%, while delays of 60, 90, and 120 years increase the loss to 5.2790%, 6.2152%, and 6.6972%, respectively. These substantial welfare losses indicate that although dynamically optimal policies are adopted before the threshold and therefore still place the economy toward the Expanding Cosmos, remaining within the policy window does not make delay costless; rather, postponement allows population decline to continue, alters the initial state for the subsequent optimal policy, and significantly lowers welfare relative to immediate implementation. In a word, the policymaker should adopt dynamically optimal fertility policy as early as possible, as timing is crucial not only for avoiding Empty Planet, but also for preventing substantial welfare losses within the policy window.

[12] See Appendix A.3 for the details.

[13] We use 30 years as an approximate generation length, consistent with the mean age of mothers at first birth in Japan averaged 30.8 years (Data Source: Ministry of Health, Labour and Welfare, Vital Statistics of Japan).

4.5 Sensitivity analysis: knowledge-production parameters

Considering that the optimal fertility subsidy internalizes the externality associated with the *idea value of people*, which is fundamentally governed by the knowledge-production parameters λ and β, we conduct a sensitivity analysis to alternative values of these parameters. We examine how sensitive our main findings are to these key parameters through two exercises: (1) Set $\lambda = 1$ and $\beta = 3.5$, which yields a lower degree of increasing returns in idea production (i.e., $\gamma \equiv \lambda/\beta$) than in the benchmark, while $\lambda = 1$ eliminates duplication externalities; (2) Set β to unity to eliminate intertemporal knowledge spillovers and λ to the median value of 0.5. This parameter set indicates a larger degree of increasing returns with ideas.

4.5.1 Small increasing returns with ideas

In the first case, the number of newly discovered ideas $\dot{A}$ is proportional to the population size, and similar to the benchmark case, ideas are getting harder to find (i.e., $\dot{A} = N_t A_t^{1-\beta}$). The steady-state values are displayed in the second column of Table 4.

Compared to the benchmark case, the high steady state rate of population growth is lower at 0.71% and its corresponding optimal fertility subsidy rate becomes smaller at 30.5%. The intuition is as follows. A lower degree of increasing returns associated with ideas suggests a smaller externality of the *idea value of people*, which reduces the gap between the equilibrium allocation and the Expanding Cosmos, requiring a lower fertility subsidy rate. The smaller population growth rate implies slower growth of technology, resulting in a higher level of knowledge per person x_h^* (i.e., 491.3). Similarly, the values of knowledge per person for the unstable steady state x_m^* also become larger, reaching a level of 801. This indicates that the threshold at which the social planner must adopt the fertility policy is widened. However, the policy window shortens from 150 to approximately 102 years because x_t reaches the threshold more quickly under this parameterization. Panel C of Table 4 shows that, although the parameter values affect both the time available for policy adoption and the magnitude of the welfare cost of delayed adoption, the cost still increases with the length of postponement.

Figure 7 reports the transition dynamics of key variables when the initial value of knowledge per person is $x_0 = 200$. Although the paths of key variables remain qualitatively similar to those in the benchmark, the welfare loss from not implementing the first-best fertility policy becomes larger, reaching 0.3252%. The loss rises further to 0.745% when x_0 is 100, as reported in the second column of Table 4. This larger welfare loss stems from a wider gap between the first-best and constrained optimal subsidy rate. However, when x_0 exceeds x_h^*, the gap narrows between the dynamically optimal fertility subsidy rate and the high steady-state optimal fertility subsidy rate, reducing the welfare loss to 0.0502% when the initial state of the economy is located at the midpoint of high and middle steady states. Figure 8 displays the corresponding transition dynamics in this case (i.e., $x_0 = 646.2$).

Table 4: Comparison of Steady State Values and Welfare Loss

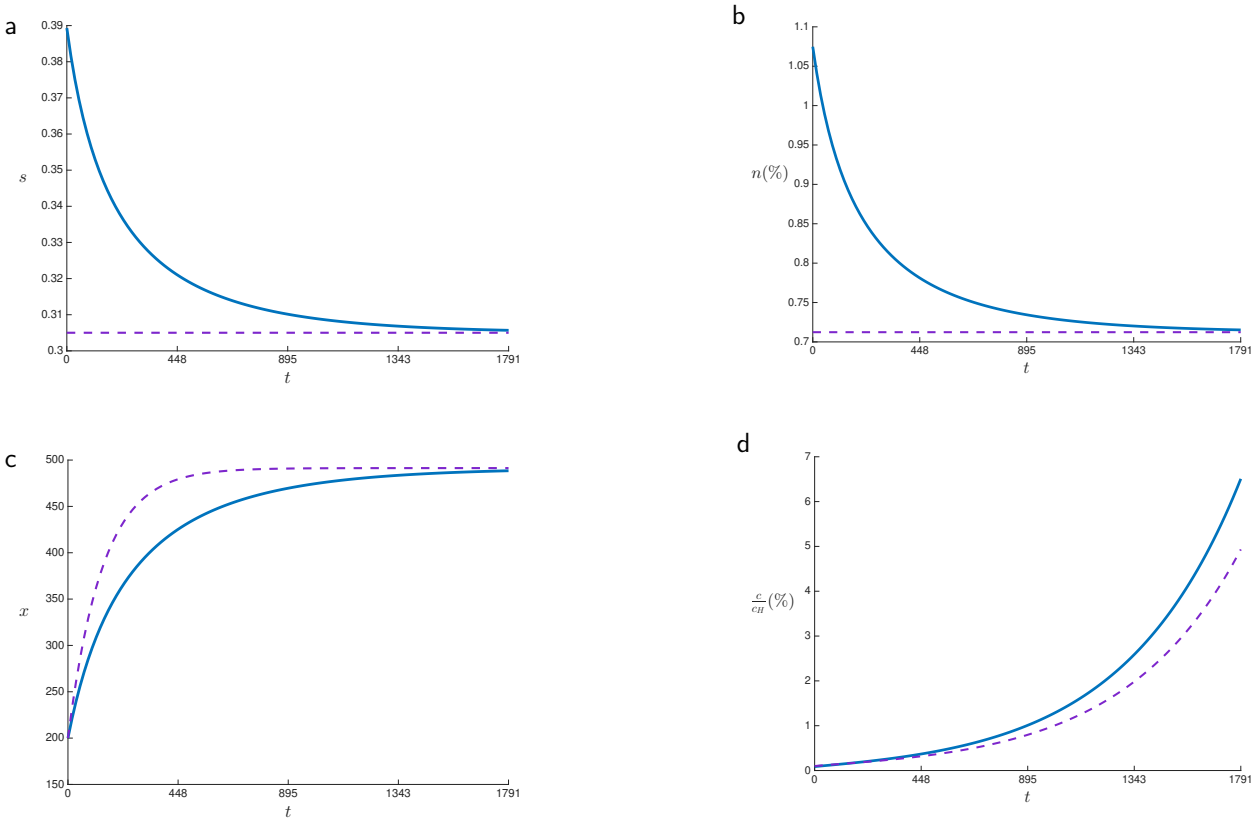
	Benchmark	$\lambda = 1, \beta = 3.5$	$\lambda = 0.5, \beta = 1$
Panel A. Steady-state values			
n_{sp}^h	1.01%	0.71%	1.09%
x_h^*	264.5	491.3	183.3
g_A^h	0.38%	0.20%	0.55%
s_h^{opt}	37.38%	30.50%	39.32%
n_{sp}^m	0.35%	0.44%	0.41%
x_m^*	754.7	801.0	489.4
g_A^m	0.13%	0.12%	0.20%
s_m^{opt}	22.15%	24.09%	23.44%
Panel B. Welfare loss of implementing the constrained optimal policy			
(i) $x_0 = 200$	0.0497%	0.3252%	0.0102%
(ii) $x_0 = 100$	0.4511%	0.7450%	0.3989%
(iii) $x_0 = \frac{x_h + x_m}{2}$	0.3829%	0.0502%	0.5512%
(iv) $x_0 = \lfloor x_m - 0.1 \rfloor$	1.1021%	0.3300%	1.5013%
Panel C. Welfare cost of delayed policy adoption			
T_{window}	150	102	157
$t = 30$	3.4803%	3.0300%	3.0093%
$t = 60$	5.2790%	4.5054%	4.6602%
$t = 90$	6.2152%	5.2235%	5.5734%
$t = 120$	6.6972%	*n/a*	6.0792%

4.5.2 Large increasing returns with ideas

Next, we analyze the second case where the idea production function is given by $\dot{A}_t = N_t^\lambda$, and the discovery of new ideas is independent of the level of technology A_t. The steady-state values and welfare losses are summarized in the third column of Table 4.

With a higher degree of increasing returns on ideas γ, the high steady state population growth rate becomes larger at 1.09% due to a stronger externality of the *idea value of people*. As a result, the long-run optimal fertility subsidy rate increases to 39.32%, and the high steady-state level of knowledge per person x_h falls to 183.3, which is below the level of knowledge per capita (i.e., $x_0 = 200$) observed in the Japanese economy. That is, when the social planner implements fertility policies in the current state, unlike the benchmark case, the dynamically optimal fertility subsidy would increase over time to stimulate the economy toward the stable Expanding Cosmos,

Fig. 7. Transition Dynamics for Variables of Interest ($\lambda = 1, \beta = 3.5$)

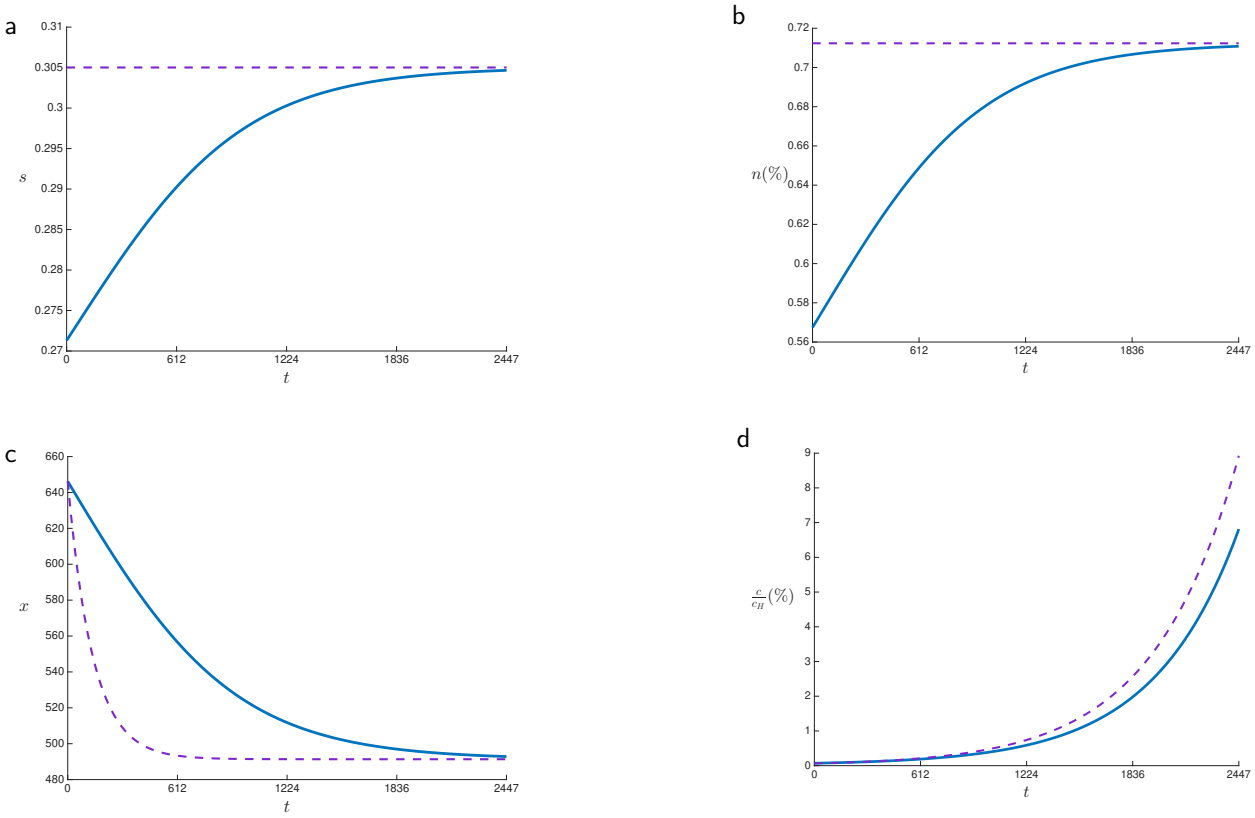


Note: Scenario (i): $x_0 = 200 < x_h$. The blue solid lines: first-best solution and the purple dashed lines: constrained optimal policy.

as shown in panel (a) of Figure 9. In this scenario, the welfare loss from not implementing the dynamically optimal fertility subsidy is only 0.0102%, since the dynamically optimal fertility subsidy rate is close to the high steady-state fertility subsidy rate. However, if fertility policy is implemented when the economy moves further from the high steady state (i.e., $x_0 = 336.4$), the trajectories of the variables would be similar to those in the benchmark case (as shown in Figure 10). The corresponding welfare loss from implementing the constrained long-run optimal fertility subsidy remains relatively large, approximately 0.5512%. In addition, although $x_0 = 200$ gets closer to the threshold x_m^* under this parameterization, the policy window lengthens slightly to 157, because the lower values of β and λ slow the evolution of x_t when the fertility rate of 2.3% is continuously adopted.

In summary, alternative values of λ and β affect both the magnitude of the dynamically optimal fertility subsidy and the steady-state values, thereby altering the welfare loss incurred by fixing the subsidy rate at its long-run level. Shifting x_h^* relative to the initial economic state can even reverse the optimal adjustment trajectory of the subsidy. Nevertheless, the key insight remains unchanged: the appropriateness of implementing a time-invariant constrained optimal fertility policy depends on the state of the economy when the social planner takes action. If

Fig. 8. Transition Dynamics for Variables of Interest ($\lambda = 1, \beta = 3.5$)



Note: Scenario (ii): $x_0 = 646.2 > x_h$. The blue solid lines: first-best solution and the purple dashed lines: constrained optimal policy.

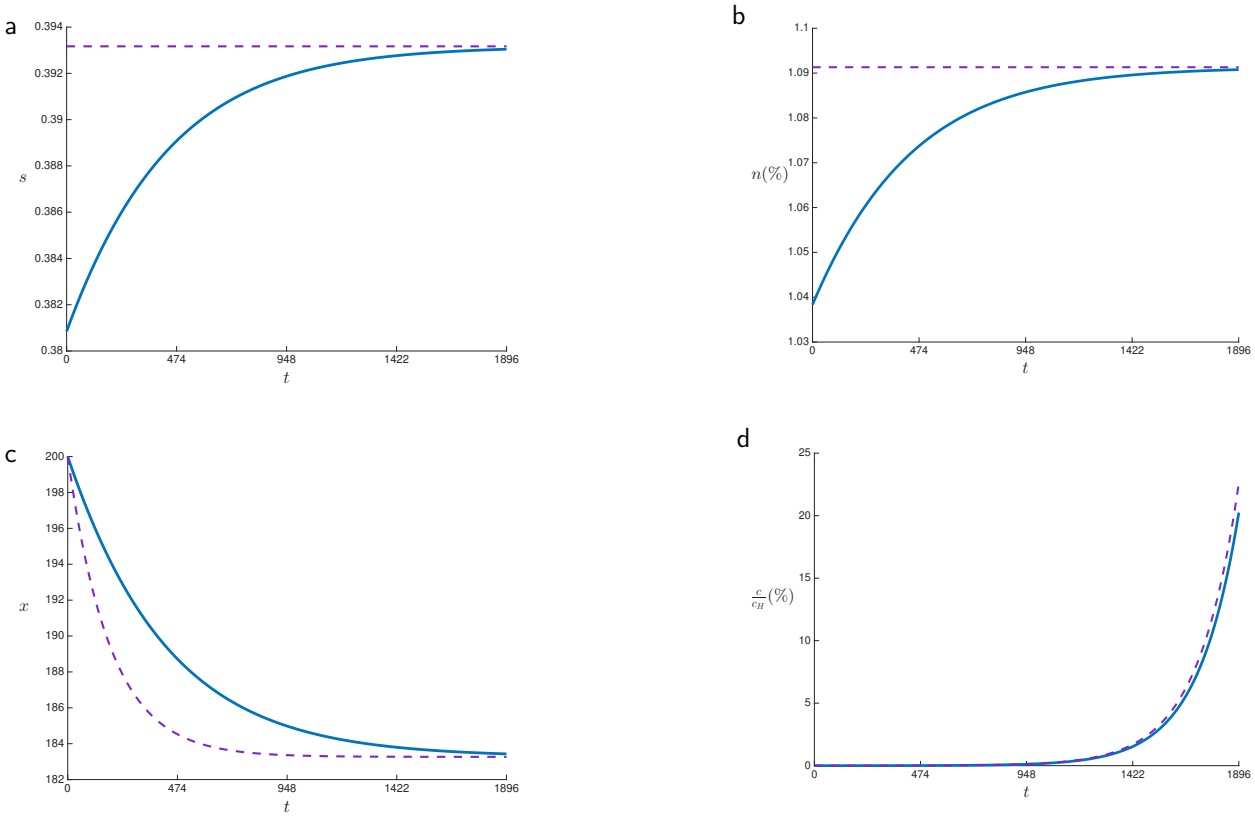
the current optimal fertility subsidy is only slightly different from the high steady-state fertility subsidy, adopting a time-invariant long-run optimal fertility subsidy can be convenient for policymakers. If the discrepancy is large, however, policymakers may need to adjust the subsidy dynamically according to the state of the economy to avoid non-negligible welfare losses. Moreover, while variations in λ and β influence the duration of Japan's policy window and the magnitude of the welfare loss, the cost of postponement increases monotonically with the delay, provided that adoption occurs within the window.

5 Labor income taxation

So far, we have discussed the scenario in which the time-variant subsidization on fertility is financed by a lump-sum tax. We now assume that fertility subsidization is financed by a linear tax on labor income, denoted as $\tau_{w,t}$. In this case, the budget constraint of the representative household in equation (3) is modified to

$$c_t = (1 - \tau_{w,t}) w_t (1 - l_t) + s_t w_t l_t. \tag{22}$$

Fig. 9. Transition Dynamics for Variables of Interest ($\lambda = 0.5, \beta = 1$)



Note: Scenario (i): $x_0 = 200 > x_H$. The blue solid lines: first-best solution and the purple dashed lines: constrained optimal policy.

Accordingly, the equilibrium allocation to fertility becomes

$$l_t = 1 - \frac{1 - \tau_{w,t} - s_t}{bV_t}, \tag{23}$$

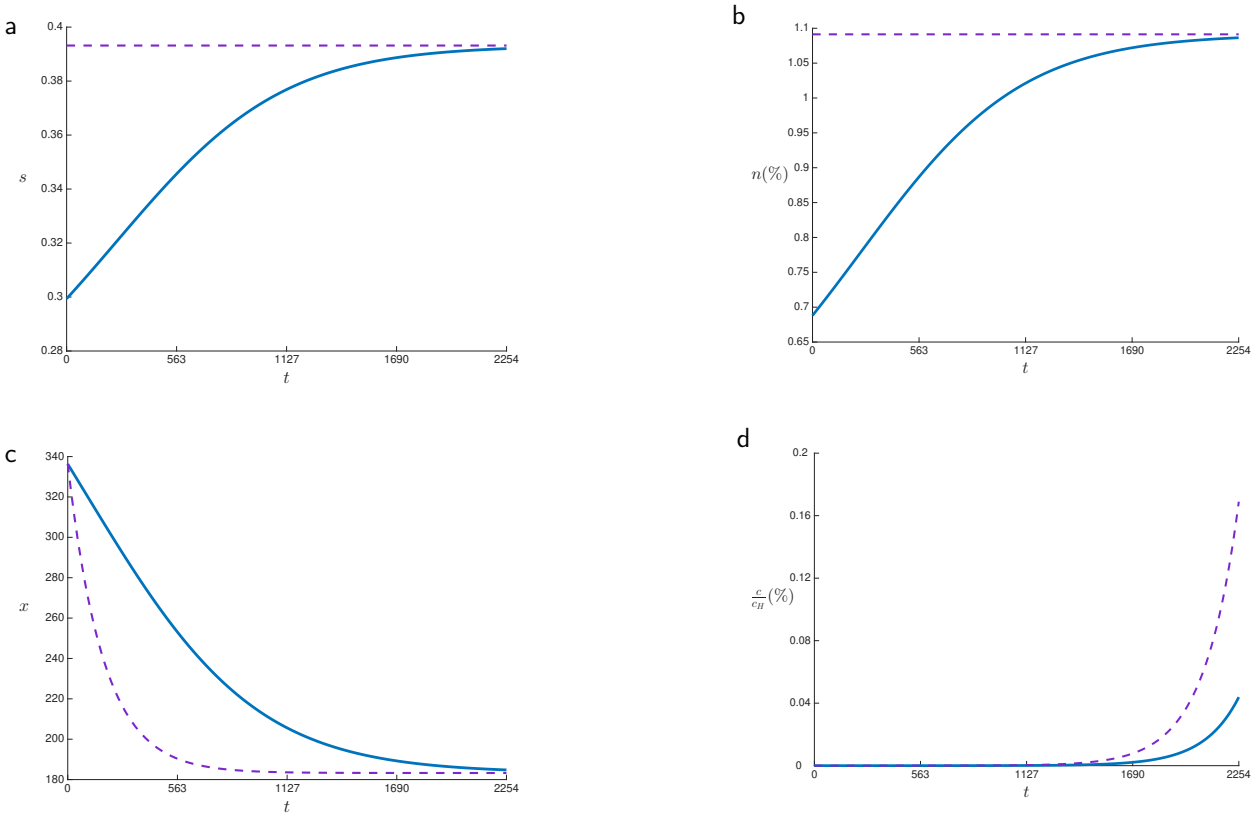
which implies that a higher rate of labor income tax reduces labor supply, thereby increasing the time allocated to child-rearing. Combining equations (8) and (23), the population growth rate under labor income taxation is given by

$$n_{eq}^{**} = b - \frac{\rho}{\epsilon} - \delta + (\tau_{w,t} + s_t)\frac{\rho}{\epsilon}, \tag{24}$$

indicating that a fertility subsidy s_t collected from the labor income tax raises the fertility rate by $(s_t + \tau_{w,t})\rho/\epsilon$. Compared to equation (9), the labor income tax generates an extra positive effect on the fertility rate, since the labor income tax reallocates the time resource from manufacturing to reproduction.

Moreover, to keep the government budget constraint balanced, the tax revenue per capita, $\tau_{w,t} w_t (1 - l_t)$, equals the government fertility subsidization, $s_t w_t l_t$, in each period. This implies

Fig. 10. Transition Dynamics for Variables of Interest ($\lambda = 0.5, \beta = 1$)



Note: Scenario (ii): $x_0 = 336.4 > x_h$. The blue solid lines: first-best solution and the purple dashed lines: constrained optimal policy.

that the tax/subsidy instrument cannot be set independently. That is, given the fertility subsidy rate s_t, the labor income tax rate $t_{w,t}$ is set at the level of $s_t l_t/(1-l_t)$ to balance the government budget.

Analogous to the previous analysis in the baseline model, using the market equilibrium with labor income tax, the dynamically optimal fertility subsidy scheme is summarized by the following proposition.

Proposition 2. *Under labor income taxation, the optimal path of population growth can be supported by setting a time-variant fertility subsidy s_t, where the optimal fertility subsidy, denoted as s_t^{opt}, satisfies:*

$$n_t = b - \frac{\rho}{\epsilon} - \delta + (s_t^{opt} + \tau_{w,t})\frac{\rho}{\epsilon}, \tag{25}$$

$$\tau_{w,t} = \frac{n_t + \delta}{b - n_t - \delta} s_t^{opt}, \tag{26}$$

where n_t evolve according to equations (19) *and* (20).

Moreover, the long-run optimal fertility subsidy is given by

$$(s^{opt})^{**} \equiv s_h^{opt} = \left(1 - \frac{n^*_{sp\{h\}} + \delta}{b}\right) \frac{\epsilon}{\rho} \left[n^*_{sp\{h\}} - (b - \rho/\epsilon - \delta)\right]. \tag{27}$$

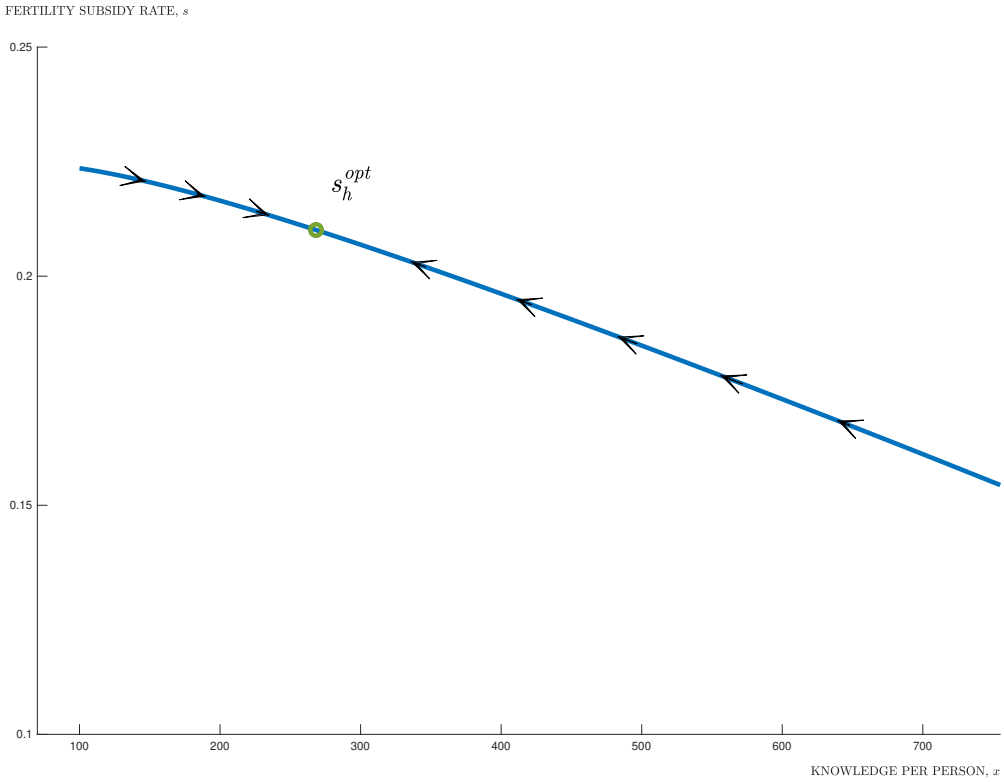
Proof. The dynamics of the optimal allocation in this new taxation scheme follow the counterpart in Proposition 1, as demonstrated by equations (19) and (20). Similarly, the externality associated with the *idea value of people* necessitates a dynamical fertility subsidy to align the equilibrium fertility rate with the first-best rate. However, when the fertility subsidy is financed by labor income taxation, the fertility in market equilibrium becomes larger, as shown by equation (25). Furthermore, since the setting of labor income tax rates is linked to the dynamic fertility subsidy through the balanced government budget constraint given by equation (26), we can derive the long-run optimal fertility subsidy, as shown in equation (27). This equation suggests that, all other things being equal, the long-run optimal fertility subsidy is lower under labor income taxation than under lump-sum taxation. □

Similar to the numerical analysis in the lump-sum tax case, we can solve for the transition dynamics for the optimal fertility subsidy quantitatively according to the dynamical system in Proposition 2, as shown in Figure 11. This figure depicts the evolution trajectory of the optimal fertility subsidy rate s_t^{opt} with respect to knowledge per person x_t under labor income taxation. The long-run optimal fertility subsidy is found to be 21%, which is significantly lower than the 37.38% observed in the lump-sum tax case. This result is consistent with equation (27). The intuition is as follows. In contrast to the case of lump-sum taxation, the fertility subsidy under labor income taxation affects the equilibrium fertility rate through two channels. First, it directly stimulates individuals' fertility choice by lowering the opportunity cost on child-rearing. Second, the fertility subsidy indirectly raises the equilibrium fertility rate through the government budget constraint. Specifically, the wage income tax, which adjusts with the rate of fertility subsidies, reallocates labor from manufacturing to reproduction. As a result, the additional positive effect arising from the labor income tax reduces the optimal fertility subsidy needed to replicate the first-best fertility rate. This reduction in the optimal fertility subsidy rate is not only evident in the long run but is also present throughout the entire transition dynamics.

Moreover, Figure 11 suggests that if the long-run optimal fertility subsidy is implemented precisely at $x_h^{'} = 268.3$, the economy will directly switch to the Expanding Cosmos; otherwise, a dynamically optimal fertility subsidy is necessary. Specifically, if the optimal fertility subsidy is adopted earlier (i.e., before the economy reaches the high steady-state knowledge level $x_h^{'}$), the government should implement decreasing fertility subsidies to gradually adjust the equilibrium population growth rate to the high steady-state rate. Conversely, if the optimal fertility subsidy is adopted later (i.e., after the economy has reached the high steady-state knowledge level $x_h^{'}$), an increasing fertility subsidy rate will be required to guide the economy back toward the Expanding Cosmos.

To analyze the implementation of the constrained long-run optimal fertility policy compared to the dynamically optimal fertility policy, the insights from the previous subsection also apply

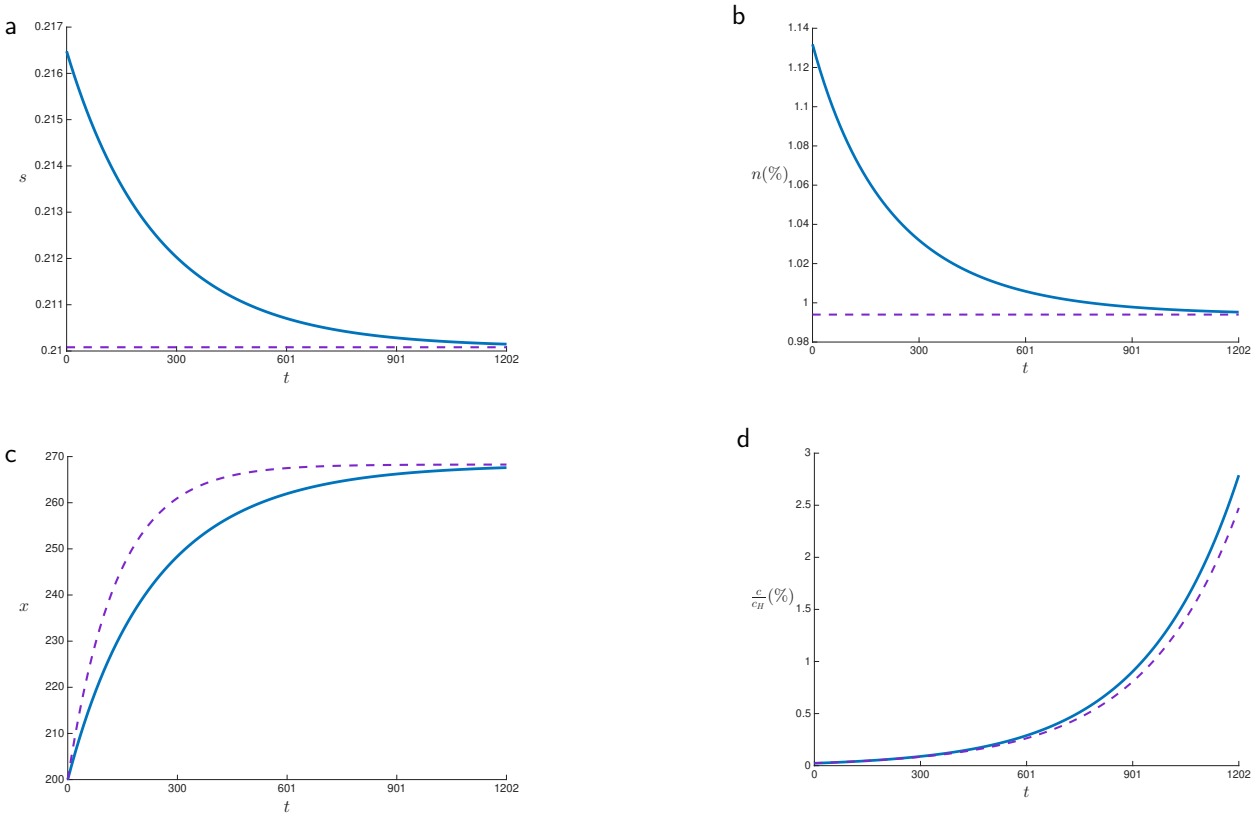
Fig. 11. Optimal Path for Fertility Subsidy Rate (under Labor Income Taxation)



to the case of labor income taxation. First, the time path for variables of interest in this new tax scheme closely resembles those in the case of the lump-sum tax, as illustrated in Figures 12 and 13. Second, the magnitude of welfare loss from implementing the constrained long-run optimal fertility subsidy rather than the dynamically optimal fertility policy is highly determined by the economy's distance from the Expanding Cosmos. In scenarios where the economy begins with low initial knowledge per person (i.e., $x_0 = 200$), the welfare loss is 0.056%. Despite a small variation in the optimal fertility subsidy rate, the welfare loss is slightly higher than the one observed under lump-sum taxation (i.e., 0.0497%), primarily because the initial state is farther away from Expanding Cosmos. In contrast, when the economy originates midway between the high and middle steady states (i.e., at 497.7), the welfare loss from not implementing the dynamically optimal fertility subsidy is relatively lower, approximately equal to 0.3459%.

Finally, we also evaluate the welfare cost of postponing to implement dynamically optimal policy. Given the initial condition $x_0 = 200$, the policy window shrinks slightly to 144 years. Within this window, longer delays in adopting the dynamically optimal subsidy lead to substantially greater welfare losses relative to immediate implementation. Specifically, delays of 30, 60, 90, and 120 years generate consumption-equivalent welfare losses of 3.4398%, 5.2121%, 6.1307%, and 6.6009%, respectively. Thus, although the labor income tax reduces the subsidy required to achieve the first-best allocation, it does not eliminate the severe welfare costs associated with policy postponement.

Fig. 12. Transition Dynamics for Variables of Interest (under Labor Income Taxation)



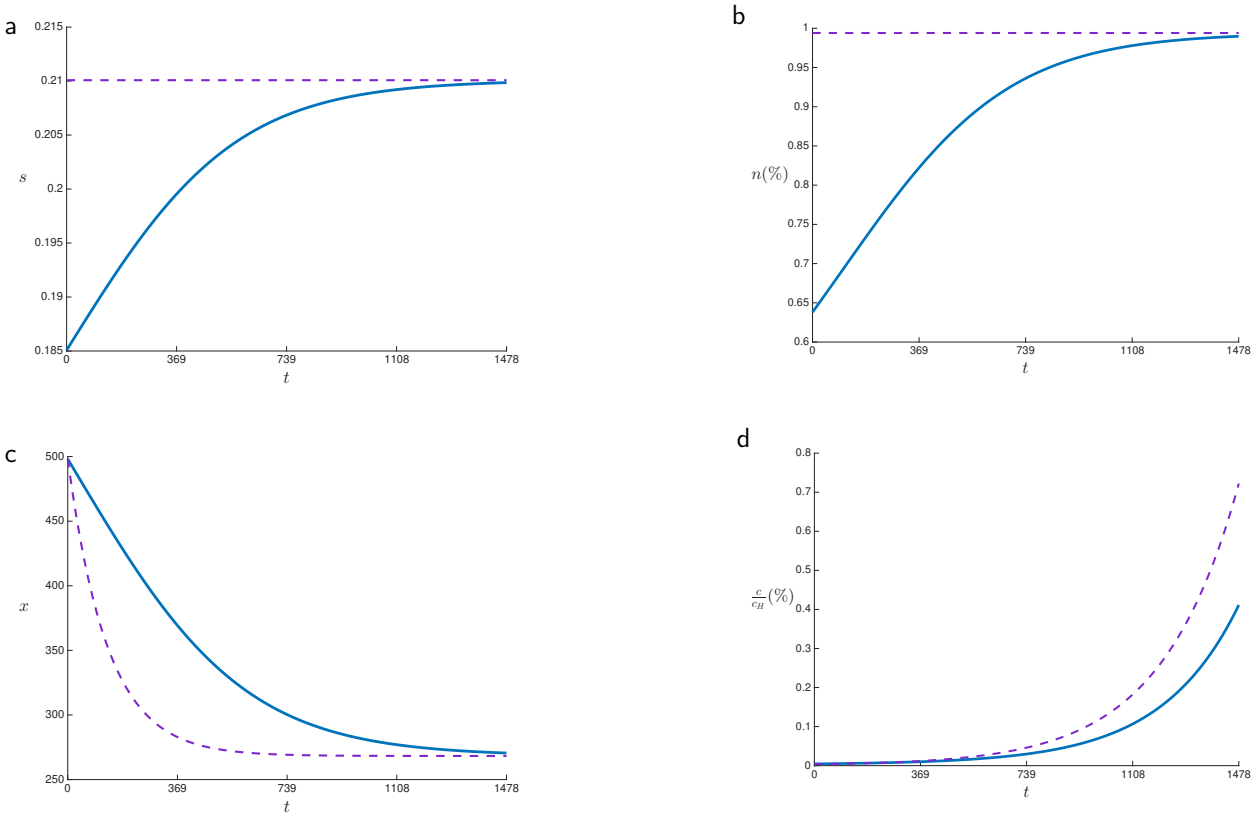
Note: Scenario (i): $x_0 = 200 < x_{\hat{H}}$. The blue solid lines: first-best solution and the purple dashed lines: constrained optimal policy.

6 Conclusion

In this study, we characterize the optimal trajectory of dynamic fertility subsidization in a semi-endogenous growth model with an endogenous, negative population growth rate. Taking the transition path into account, we compare the welfare difference between implementing the first-best fertility policy and the constrained optimal long-run counterpart, while quantifying the cost of delayed policy adoption. We first theoretically depict the dynamics of the optimal fertility subsidy by analyzing the decentralized equilibrium featuring a negative population growth rate and the optimal allocation with three steady states. To avoid the Empty Planet trap, timely adoption of the fertility subsidy is essential. Moreover, to achieve the Expanding Cosmos, the social planner must implement a time-variant optimal fertility subsidization.

Using aggregate data from Japan, we conduct a quantitative analysis to explore the transitional dynamics of the fertility subsidy rate. Our results indicate that the current fertility subsidy rate of 2.3% is insufficient to prevent the Japanese economy from entering the Empty Planet; a higher subsidy rate (i.e., 13.96% in the benchmark case) is necessary and should be implemented before the economy reaches the threshold. To reach the Expanding Cosmos, a higher dynamic

Fig. 13. Transition Dynamics for Variables of Interest (under Labor Income Taxation)



Note: Scenario (i): $x_0 = 497.7 > x_{\mathit{ft}}$. The blue solid lines: first-best solution and the purple dashed lines: constrained optimal policy.

fertility rate is required. Furthermore, we find significant variations in the optimal fertility subsidy rate, with substantial differences in the time paths of key variables between the first-best and constrained policies. As a result, the magnitude of welfare loss from implementing the time-invariant long-run optimal fertility subsidy is highly influenced by the economy's distance from the Expanding Cosmos. If the initial state of the economy is close to the targeted steady state, the welfare loss can be quite small; otherwise, the welfare loss is non-negligible. Last but not least, we find that while the current subsidy keeps Japan within the policy window for approximately 150 years, postponing the first-best policy generates progressively larger welfare losses.

These results concerning dynamically optimal fertility subsidization are independent upon whether subsidies are financed by a lump-sum tax or a labor income tax. However, when the subsidy is funded through a tax on labor income, the dynamically optimal subsidy needed to replicate the first-best fertility rate becomes smaller. The fiscal feasibility of a large-scale fertility subsidy is a salient concern in the context of significant budget deficits. However, our quantitative framework is not intended to be prescriptive regarding specific subsidy rates; instead, it seeks to quantify the social return of future innovation generated by additional births. The results underscore the importance of promptly augmenting current fertility support, subject to fiscal

constraints, to forestall the Empty Planet outcome precipitated by negative population growth.

This study provides a valuable framework for theoretically investigating the impact of fertility subsidies in the context of declining population. However, to maintain tractability, some important factors are omitted, such as capital accumulation (both human and physical capital), a monopolistic intermediate-good sector, and the nuances of demographic transition. Therefore, future research should be conducted in alternative models with richer setups, providing a more comprehensive and robust understanding of fertility subsidy. Additionally, various pronatal policies exist beyond mere cash allowances. Countries often implement a combination of measures, including maternity and paternity leave, childcare services, and housing subsidies. Extending the current analysis by considering other policies or a policy mix could be promising directions for future research.

A Appendix

A.1 Numerical solution of transition dynamics of optimal fertility subsidy

We first solve for the optimal path using the reverse-shooting method employed by Jones (2022). Specifically, we identify the high steady state and middle steady state. The high steady state is a saddle point, which features a positive eigenvalue and a negative one. To trace its stable manifold, we move a tiny distance away from the high steady state in the direction of the eigenvector associated with the negative eigenvalue and integrate the full nonlinear system backward. We repeat the backward integration from two nearby points on opposite sides of the high steady state. One starting point generates the branch with $x_t < x_h^*$, while the other generates the branch with $x_h^* < x_t < x_m^*$. Then, given that fertility rate n_t is a control variable and may jump when the policy is introduced, we can recover the dynamically optimal subsidy s_t^{opt} from the decentralized fertility condition for any initial state $x_0 < x_m^*$. Additionally, the transition system is solved using MATLAB's adaptive Runge–Kutta solver ode45 with relative and absolute tolerances set to 10^{-10}.

A.2 Measuring welfare loss from constrained optimal policy

Using the state variable $x_t \equiv A_t^{\beta}/N_t^{\lambda}$ and $p_t = \ln N_t$, the utility function (1) can be written as:

$$U_0 = \int_0^{\infty} e^{-\rho t}\left[(\sigma/\beta)\ln x_t + \ln(1-l_t) + (\epsilon+\gamma)p_t\right]dt, \quad N_0 = 1. \tag{A.1}$$

We measure the welfare cost of fixing the fertility subsidy at its long-run optimal level by comparing two allocations starting from the same initial state. Under the first-best policy, the dynamically optimal subsidy s_t^{opt} replicates the planner's transition path obtained from the reverse-shooting procedure described above. We denote the associated allocation by x_t^{fb}, ℓ_t^{fb}, and p_t^{fb}. Under the constrained policy, the subsidy is fixed at its high-steady-state level s_h^{opt}, so that:

$$p_t^{co} = n_{sp}^h t, \qquad \ell_t^{co} = \ell_{sp}^h, \qquad x_t^{co} = x_h + (x_0 - x_h)e^{-\lambda n_{sp}^h t}. \tag{A.2}$$

To calculate the total welfare difference, we first compute the cumulative flow-utility difference between the two policy regimes. Let D_t denote the welfare advantage of the first-best policy over the constrained policy accumulated up to time t. It evolves according to

$$\dot{D}_t = e^{-\rho t}\left[\frac{\sigma}{\beta}\ln\left(\frac{x_t^{fb}}{x_t^{co}}\right) + \ln\left(\frac{1-\ell_t^{fb}}{1-\ell_{sp}^h}\right) + (\epsilon+\gamma)\left(p_t^{fb} - p_t^{co}\right)\right], \qquad D_0 = 0. \tag{A.3}$$

Taking the limit of the cumulative welfare gap as t approaches infinity gives the lifetime welfare difference between the two policies $D = U_{fb} - U_{co}$. In the numerical implementation, the remaining welfare difference after the computed paths enter a sufficiently small neighborhood of the high steady state is captured by the corresponding continuation value. Following Chu and Furukawa (2011), the welfare difference can be expressed as a consumption-equivalent welfare

loss, defined as $\Delta c \equiv 1 - \exp(-\rho D)$. Δc represents the magnitude of the hypothetical loss in steady-state per capita consumption from not implementing the first-best subsidy.

A.3 Measuring welfare cost of delayed adoption

We measure the welfare cost of delayed adoption by comparing immediate implementation of the dynamically optimal subsidy with implementation after a delay of d years. We restrict the adoption delay to the policy window, $0 \leq d < T_{window}$, where T_{window} is defined as the first time at which the economy reaches the threshold x_m^* if the fertility subsidy rate remains at the current fertility subsidy (i.e., $x^{eq}_{T_{window}} = x_m^*$). With the fertility subsidy held at its current rate at 2.3%, the equilibrium population growth rate remains constant at n_{eq}^*, and knowledge per person evolves according to $\dot{x}_t = \beta - \lambda n_{eq}^* x_t$, which implies that:

$$x_t^{eq} = \frac{\beta}{\lambda n_{eq}^*} + \left(x_0 - \frac{\beta}{\lambda n_{eq}^*} \right) e^{-\lambda n_{eq}^* t}. \tag{A.4}$$

Setting $x^{eq}_{T_{window}} = x_m^*$, we can obtain:

$$T_{window} = \frac{1}{-\lambda n_{eq}^*} \ln \left[\frac{\beta - \lambda n_{eq}^* x_m^*}{\beta - \lambda n_{eq}^* x_0} \right]. \tag{A.5}$$

Under the benchmark parameterization, this expression yields $T_{\text{window}} = 150.2$ years, which we report as approximately 150 years.

During the delay, the economy remains on the decentralized equilibrium path implied by the current subsidy. The discounted welfare accumulated before policy adoption is

$$U^{pre}(d) = \int_0^d e^{-\rho t} \left[\frac{\sigma}{\beta} \ln x_t^{eq} + \ln(1 - \ell_{eq}) + (\epsilon + \gamma) p_t^{eq} \right] dt, \tag{A.6}$$

where $p_t^{eq} = n_{eq}^* t$ and $x_t^{eq} = \beta/(\lambda n_{eq}^*) + [x_0 - \beta/(\lambda n_{eq}^*)] e^{-\lambda n_{eq}^* t}$.

Let $V^*(x, p)$ denote the planner's continuation value when the dynamically optimal policy is implemented at state (x, p). The social planner's problem implies that

$$V^*(x, p) = W(x) + \left(\frac{\epsilon + \gamma}{\rho} \right) p, \tag{A.7}$$

where $W(x)$ is the component of the planner's continuation value associated with state x along the optimal path. When the first-best policy is adopted at date d, the economy reaches the state (x_d^{eq}, p_d^{eq}), and the continuation value at time d is therefore: $V^*(x_d^{eq}, p_d^{eq}) = W(x_d^{eq}) + [(\epsilon + \gamma)/\rho] p_d^{eq}$. Consequently, lifetime welfare under delayed adoption is

$$U^{da}(d) = U^{pre}(d) + e^{-\rho d} \left[W(x_d^{eq}) + \left(\frac{\epsilon + \gamma}{\rho} \right) p_d^{eq} \right]. \tag{A.8}$$

This equation implies that delayed-adoption welfare contains two components: the flow utility generated under the current policy before adoption and the entire continuation value generated by the dynamically optimal policy after adoption.

Under immediate adoption, lifetime welfare is given by the planner's value at the initial state:

$$U^{im} = V^*(x_0, 0) = W(x_0). \tag{A.9}$$

The welfare cost of delaying adoption by d years is thus defined as: $D^{da}(d) = U^{im} - U^{da}(d)$. As in the previous subsection, we express this welfare difference as a consumption-equivalent welfare loss: $\Delta c^{da}(d) = 1 - e^{-\rho D^{da}(d)}$.

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